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POTENTIAL FLOW ABOUT A PROLATE SPHEROID IN AXIAL HORIZONTAL MOTION BENEATH A FREE SURFACE

by

César Farell

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mechanics and Hydraulics In the Graduate College of The University of Iowa

August, 1968

Chairman: Professor Louis Landweber

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This dissertation is hereby approved as a creditable report on research carried out and presented in a manner which warrants its acceptance as a prerequisite for the degree for which it is submitted. It is to be understood, however, that neither the Department of Mechanics and Hydraulics nor the dissertation adviser is responsible for the statements made or opinions expressed.

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An Abstract

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The potential flow about a prolate spheroid in axial horizontal motion beneath a free surface is treated analytically. While the free-surface boundary condition is linearized, the boundary condition on the surface of the body is satisfied exactly. Thus, an "exact" solution, within the theory of infinitesimal waves, is obtained. The solution is sought in the form of a distribution of sources on the surface of the spheroid, of unknown density; the analysis yields an infinite set of equations for determining the coefficients of the expansion of the density function in spherical harmonics (and therefore for determining the coefficients of the expansion of the potential in spheroidal harmonics). An expression is derived for the wave resistance of the spheroid in terms of these coefficients through application of the Lagally theorem. The expression for the wave resistance given by Havelock in 1931 is obtained as the first approximation in the present analysis.

Numerical evaluations were performed using an IBM 360/65 computer, for a Froude number (defined with respect to the distance between foci) of 0.4, a focal distance equal to twice the depth of submergence, and several values of the eccentricity. The numerical solution of the infinite set of equations was obtained keeping an increasing number of equations (and, therefore, calculating an increasing number of coefficients of the series expansions), up to a maximum of twenty. The wave resistance and the density of the source distribution were evaluated at each stage, the latter along meridian planes of the spheroid. For a prolate spheroid with a slenderness ratio slightly larger than five the wave resistance is larger than Havelock's by about 34%. For

slenderness ratios of 3.64 and 2.40 the corrections are as much as 90% and 368%, respectively, of Havelock's approximation (the spheroid corresponding to the latter slenderness ratio is very close to piercing the free surface).

An infinite set of equations, essentially equivalent to that obtained in this work for determining a source distribution on the surface of the spheroid which satisfies exactly the boundary condition on its surface, was obtained by Bessho using an entirely independent derivation. The coefficients of Bessho's system of equations, however, appear to be incorrect, possibly because of typographical errors, and his numerical evaluations are rather inaccurate. The value of the wave resistance obtained by Bessho, for a Froude number of 0.395, a focal distance equal to twice the depth of submergence, and a slenderness ratio of 4.17, exceeds Havelock's approximation by 146%; according to the numerical evaluations reported here, the correction should instead be in the neighborhood of 60 to 65% of Havelock's value.

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INTRODUCTION

Even the simplest problems involving free surface waves are difficult to treat analytically when formulated exactly. In dealing with these problems, therefore, one must introduce simplifying assumptions into the equations of motion and the boundary conditions, and replace the original problem by a new one which is more amenable to mathematical treatment and which should approximate the original one in some prescribed way. If viscosity is neglected and irrotational flow is assumed, the problem reduces to finding solutions of the Laplace equation. The free-surface boundary condition, however, is nonlinear, even if surface tension is neglected, and moreover it must be applied on a surface of unknown location. As a result, further simplifications are still needed in most cases. For flows about submerged obstacles, the assumption of irrotational flow may lead to a new problem which does not approximate the original one; such is the case, for example, with flow about circular cylinders and spheres, these two shapes being very attractive to the mathematical researcher because of the simplifications they bring about in the mathematical expressions.

In order to deal with the nonlinear free-surface boundary condition, special approximation techniques have been developed. For flows about submerged bodies, these techniques transform the problem into a series of linear problems with nonhomogeneous boundary conditions (with the

exception of the first order or infinitesimal wave approximation, for which the corresponding boundary condition is homogeneous), to be applied on the plane of the undisturbed free surface. Even within the theory of infinitesimal waves, however, an exact solution is difficult to obtain. An additional approximation is involved in many researches, since the boundary condition on the surface of the body is not satisfied exactly. If one assumes that the shape of the body is such that perfect fluid theory can yield results of practical significance, the determination of the errors due to the failure to satisfy the nonlinear free-surface boundary condition on the one hand, and the boundary condition on the surface of the body on the other, is of great practical interest.

In the present work, the wave motion produced when a stream of constant velocity is incident upon a submerged prolate spheroid with its axis parallel to the free surface and in the direction of the stream will be treated. This shape is of interest in connection with the wave resistance of submarines; moreover, perfect fluid theory can be expected to yield results of practical significance when applied to determine the flow about slender prolate spheroids. The free surface boundary condition will be linearized; the boundary condition on the surface of the body will be satisfied exactly. Thus, an answer will be provided, for this particular shape, to the latter of the two problems posed in the preceding paragraph.

A short summary of past researches dealing with the theory of infinitesimal waves in flows about submerged obstacles is included below. Only streams of infinite depth are considered.

The first detailed evaluation of the wave motion caused by a fully submerged obstacle is contained in a paper by Lamb (1913) in which the disturbance produced in the flow of a uniform stream (of infinite depth) by a submerged circular cylinder is dealt with. The evaluation yields, actually, the disturbance produced in the flow of the stream by the dipole which generates the cylinder in an unbounded fluid. The boundary condition on the surface of the cylinder is not satisfied and an anomalous moment on the cylinder results (Wehausen 1960, p. 576; see also Havelock 1929). As Tuck (1965) has shown, no closed body is actually generated. Thus, Lamb's analysis provides only a first step in the solution of the linearized problem.

Let ϕ_0 be the velocity potential of a uniform (unbounded) stream. Let ϕ_1 be the velocity potential of the image of ϕ_0 in the submerged body; that is, ϕ_1 is such that $\phi_0 + \phi_1$ satisfies the boundary condition on the surface of the body. Let $\phi_1^{(s)}$ be the velocity potential of the image of ϕ_1 in the free surface; that is, $\phi_1^{(s)}$ is such that $\phi_1 + \phi_1^{(s)}$ satisfies the free-surface boundary condition and the radiation condition at $x = -\infty$. Let in general $\phi_n^{(s)}$ be the velocity potential of the image of ϕ_n in the free surface and ϕ_{n+1} be the velocity potential of the image of $\phi_n^{(s)}$ in the submerged body. The sequence

$$\phi_0 + \phi_1 + \phi_1^{(s)} + \ldots + \phi_n + \phi_n^{(s)} + \phi_{n+1} + \ldots ,$$

if convergent, yields the complete solution to the linearized problem.

This program has been carried out by Wehausen (Wehausen & Laitone 1960, pp. 574 et seqq.) for the case of the circular cylinder. In

this case, $\phi_n^{(s)}$ can be obtained from ϕ_n by means of a formula of Kochin, and ϕ_{n+1} from $\phi_n^{(s)}$ by using Milne-Thomson's circle theorem; Wehausen has shown the series to be convergent. Cylinders of other shapes can probably be handled by a combination of this technique and conformal mapping.

A complete solution for the submerged circular cylinder was also given by Havelock in 1936. In previous papers (1917, 1927, 1929) Havelock had applied to this problem the method of successive images later used by Wehausen to give a complete solution, evaluating the second set of images within the cylinder and the corresponding image in the free surface, and thus carrying the computations two stages further than in Lamb's solution. In his 1936 paper, however, he used a different approach and obtained the solution by expanding the complex potential of the system of sources and sinks within the cylinder in a Laurent series, $\Sigma A_n z^{-n}$, about the origin. The free-surface boundary condition (and the radiation condition) can then be satisfied by adding a suitable expression in terms of the coefficients A_n of the Laurent series, and the boundary condition on the surface of the body yields an infinite set of equations for the coefficients A_n. The procedure, which lends itself well to approximate computations, amounts to producing the flow outside the cylinder due to the system of singularities inside by placing at its center an infinite number of multipoles, modified to satisfy the free-surface boundary condition (and the radiation condition), and whose strength is chosen so as to satisfy the boundary condition on the surface of the body. Havelock found, for the wave resistance of the circular cylinder, the expression

$$R = 4\pi^2 \rho U^2 (k_o a)^3 \, e^{-2k_o d} \left\{ 1 - 2r_1 (k_o a)^2 - (r_2 - 3r_1^{\,2} + s^2)\,(k_o a)^4 + \ldots \right\}$$

where a is the radius of the cylinder, d the depth of submergence of its axis,

$$k_o = g/U^2,$$

$$s = 2\pi e^{-\alpha},$$

$$\alpha = 2k_o d,$$

$$r_n = \frac{n!}{\alpha^{n+1}} + 2\left\{ \frac{(n-1)!}{\alpha^n} + \frac{(n-2)!}{\alpha^{n-1}} + \ldots + \frac{1}{\alpha} - e^{-\alpha}\,\mathrm{li}(e^{\alpha}) \right\},$$

and li denotes the logarithmic integral. The first term in this expression is that obtained by Lamb for the wave resistance of the circular cylinder.

The approximation to the wave resistance of a submerged body which is obtained by evaluating the flow about the singularity distribution that produces the body in an unbounded stream has been calculated by Havelock for the sphere (1917), for prolate and oblate spheroids moving in the direction of the axis of symmetry and at right angles to it (1931a), and for an ellipsoid with unequal axes moving in the direction of the longest axis (1931b). Havelock evaluated the wave resistance of the sphere by direct integration of the horizontal component of the pressure on its surface; the method of successive images was applied to compute the second set of images within the sphere, needed in order to satisfy the boundary condition on its surface to the required degree of approximation. The computation of this second set of images can be dispensed with if Lagally's theorem is used to obtain the wave

resistance to the same degree of approximation. This fact was noticed by Havelock in his 1929 paper on the circular cylinder, in which a proof of Lagally's theorem for the steady two-dimensional case in which only sources, sinks and doublets are present is given. The proof is based on the Blasius formulas for the force and moment on a closed contour in two-dimensional, incompressible, potential flow; the name of Lagally is not associated with the result. To compute the wave resistance for the spheroid and the ellipsoid, however, Havelock made use of a previous result for the wave resistance of an arbitrary set of doublets in a uniform stream (1928; see also 1932) obtained using an artificial method due to Lamb (1926). The method consists in making use of Rayleigh's artifice of including a fictitious dissipative body force, $-\rho\mu'V$, proportional to the disturbance velocity, calculating the energy dissipation as

$$RU = \mu'\rho \int_S \phi \frac{\partial\phi}{\partial n}\, dS$$

where the integration is taken over the boundary of the fluid, and interpreting the finite limiting value of this expression, when μ' is made to approach zero, as the rate at which energy is propagated outwards in the wave motion.

In his 1929 paper on the circular cylinder Havelock pointed out also that the second set of images within the cylinder is needed if Lagally's theorem is used to verify (to the degree of approximation afforded by the first image within the cylinder and the corresponding image in the free surface) that the moment about the center of the cylinder vanishes. Indeed, application of Lagally's theorem shows

immediately that a contribution of the required order of magnitude arises from the interaction of the external uniform stream and the second set of images. For a prolate spheroid moving along its axis, the contribution to the moment arising from this interaction was calculated by Havelock (1952). The same task was undertaken by Pond (1951) for the Rankine ovoid and later (1952) for the more general case of an elongated body of revolution. Pond does not obtain the second image system exactly. Instead he uses Munk's technique and obtains, with some additional simplifications, an approximate image system in the form of a distribution of doublets along the axis of the body between the limits of the distribution that produces the body in the uniform unbounded stream. Lagally's theorem yields then immediately the required moment. Pond's result agrees with the approximation that Havelock, based upon his analysis for the prolate spheroid, proposes for long slender bodies of revolution.

The case of the fully submerged prolate spheroid, which will be dealt with in this work, was treated analytically by Bessho (1957), who tried to satisfy exactly the boundary condition on the surface of the spheroid using a distribution of sources on it. Bessho considered first an ellipsoid with three unequal axes and was thus able to use in the solution of the problem several results on Lamé's functions given by Hobson (1931, Chap. 11). His final expressions, however, appear to be incorrect, possibly because of typographical errors, and his numerical evaluations are rather inaccurate. In the present work, a more direct attack using spherical coordinates yields an equivalent set of equations for determining the source distribution, but with certain

significant corrections. Furthermore, the numerical evaluations are performed to a high degree of accuracy, avoiding the rough approximations employed by Bessho.

BASIC EQUATIONS

For convenience of reference the equations describing the irrotational motion, under gravity, of an incompressible fluid having a free surface are collected here. Derivations of these equations can be found elsewhere.

Let Oxyz be a right-handed rectangular coordinate system and let the y axis be directed vertically upwards. Since the motion is irrotational, we have

$$\bar{V} = \nabla \Phi \tag{1}$$

where $\bar{V}$ is the velocity vector and Φ is the velocity potential. Since the fluid is incompressible, it follows from the equation of continuity that Φ must satisfy Laplace's equation

$$\nabla^2 \Phi = 0 \tag{2}$$

The Euler equations of motion, on the other hand, yield the integral

$$gy + \frac{p}{\rho} + \frac{V^2}{2} = -\frac{\partial \Phi}{\partial t} \tag{3}$$

where g is the acceleration of gravity, p is the pressure, ρ is the mass density of the fluid, and t is time.

To these equations we must add the appropriate boundary conditions for the problem on hand. Let $S(t)$ be a boundary surface and let it be described by $F(x,y,z,t)=0$. Since no transfer of matter takes

place across $S(t)$, we must have at every point of the surface

$$V_x \frac{\partial F}{\partial x} + V_y \frac{\partial F}{\partial y} + V_z \frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0 \tag{4}$$

where V_x, V_y, and V_z are the components of the velocity vector $\overline{V}$. At a fixed boundary, condition (4) reduces to

$$\frac{\partial \Phi}{\partial n} = 0 \tag{5}$$

$\partial/\partial n$ denoting the derivative in the direction of the outward normal to the boundary. If

$$y = \eta(x,z,t) \tag{6}$$

describes the free surface of the fluid, condition (4) takes the form

$$V_x \eta_x - V_y + V_z \eta_z + \eta_t = 0 \tag{7}$$

Let p_o be the constant pressure above the free surface. Then, in addition to the kinematic condition (7), we must have at every point of the free surface

$$p(x,y,z,t) = p_o \tag{8}$$

since we neglect both surface tension and viscous effects. When use is made of Equation (3), this condition becomes

$$g\eta + \frac{V^2}{2} + \frac{\partial \Phi}{\partial t} = 0 \tag{9}$$

to be satisfied on $y = \eta(x,z,t)$, the additive constant having been merged in the value of $\partial \Phi/\partial t$.

In the following, the perturbation produced by a submerged obstacle

in the flow of a stream of constant velocity U will be considered. It is then convenient to write

$$\Phi = Ux + \phi \tag{10}$$

where the perturbation potential ϕ satisfies the Laplace equation

$$\nabla^2 \phi = 0 \tag{11}$$

and to let u, v, and w be the components of the perturbation velocity, that is

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z} \tag{12}$$

The boundary conditions become

$$\frac{\partial \Phi}{\partial n} = U \frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial n} = 0 \tag{13}$$

on the surface of the body, and, since the motion is steady,

$$(U+u)\eta_x - v + w\eta_z = 0 \tag{14}$$

and

$$g\eta + \frac{1}{2}\left\{(U+u)^2 + v^2 + w^2\right\} = \frac{1}{2}U^2 \tag{15}$$

on the free surface $y = \eta(x,z)$.

The boundary condition that the velocity potential must satisfy on the free surface can be obtained by eliminating η between (14) and (15), as was done by Landweber (1964). It is more convenient, however, to proceed as follows. On the free surface

$$p(x,y,z) = p_o$$

and therefore

$$\frac{Dp}{Dt} = \overline{V} \cdot \nabla p = 0 \tag{16}$$

Substituting p from (3) into (16) we obtain

$$gv + \frac{1}{2}\overline{V} \cdot \nabla V^2 = 0 \tag{17}$$

This is the nonlinear boundary condition to be satisfied by the velocity potential on the free surface. If we let

$$q^2 = u^2 + v^2 + w^2$$

since

$$V^2 = U^2 + q^2 + 2Uu$$

we can write (17) in the form

$$gv + U^2\frac{\partial u}{\partial x} + U\frac{\partial q^2}{\partial x} + \frac{1}{2}\left(u\frac{\partial q^2}{\partial x} + v\frac{\partial q^2}{\partial y} + w\frac{\partial q^2}{\partial z}\right) = 0 \tag{18}$$

or, making use of (12),

$$U^2\frac{\partial^2 \phi}{\partial x^2} + g\frac{\partial \phi}{\partial y} = -U\frac{\partial q^2}{\partial x} - \frac{1}{2}\left(u\frac{\partial q^2}{\partial x} + v\frac{\partial q^2}{\partial y} + w\frac{\partial q^2}{\partial z}\right) \tag{19}$$

to be satisfied on $y = \eta(x,z)$.

For a stream of infinite depth we also have as a boundary condition

$$\lim_{y \to -\infty} \operatorname{grad} \phi = 0 \tag{20}$$

that is, the perturbation vanishes far below the body. In addition to the boundary conditions (13), (14), (15), and (20) we must also have

$$\lim_{x \to -\infty} \operatorname{grad} \phi = 0 \tag{21}$$

that is, the motion must also vanish far ahead of the body. This so-called "radiation condition" need not be imposed if Rayleigh's artifice of introducing a fictitious dissipative body force proportional to the disturbance velocity is used (Lamb 1932, p. 399; see also Wehausen and Laitone 1960, p. 479) or if an initial value problem can be formulated (for which the boundary condition at infinity is simply that the motion vanishes everywhere) whose solution yields the steady-state solution sought as $t \to \infty$ (Stoker 1957, pp. 174-181; see also Wehausen and Laitone 1960, p. 472, for additional references).

Although Laplace's equation (11) and the boundary condition (13) on the boundary of the body are linear, the free surface boundary conditions (14) and (15) are nonlinear. This nonlinearity precludes the use of techniques of solution based on the principle of superposition. Thus, the method of separation of variables and expansion in eigenfunctions cannot be used, and neither can the method of Green's functions or singularity distributions. Moreover, the free-surface boundary conditions are rather inconvenient because they are to be applied on a surface of unknown location.

Under these conditions, special problems in the theory of surface waves have been treated by using approximation techniques (actually perturbation procedures) of which a fairly complete account has been

given by Wehausen (Wehausen and Laitone 1960). Since only the first order approximation will be treated here, the perturbation analysis applicable in our case is not included and the reader is referred to Wehausen's treatise for details of the method. The first order approximation consists of neglecting all second order terms in the free-surface boundary conditions and applying them on the plane $y = 0$ instead of on the free surface $y = \eta(x,z)$. Equations (14), (15), and (19) become, respectively,

$$U\eta_x - v = 0 \tag{22}$$

$$g\eta + Uu = 0 \tag{23}$$

and

$$U^2 \frac{\partial^2 \phi}{\partial x^2} + g\frac{\partial \phi}{\partial y} = 0 \tag{24}$$

to be satisfied on the plane $y = 0$. Equation (24) can be obtained directly from (22) and (23) by eliminating η between these two equations.

It is interesting to note the boundary condition that the second-order contribution to the perturbation potential must satisfy. This is

$$U^2 \frac{\partial^2 \phi^{(2)}}{\partial x^2} + g\frac{\partial \phi^{(2)}}{\partial y} = \frac{U}{g}\frac{\partial \phi^{(1)}}{\partial x}\frac{\partial}{\partial y}\left\{U^2\frac{\partial^2 \phi^{(1)}}{\partial x^2} + g\frac{\partial \phi^{(1)}}{\partial y}\right\} - U\frac{\partial}{\partial x}\left\{\operatorname{grad}\phi^{(1)}\right\}^2 \tag{25}$$

to be satisfied on the plane $y = 0$. Once the first-order approximation $\phi^{(1)}$ is known, therefore, the second-order contribution can be obtained by solving another linear problem with a nonhomogeneous boundary condition on the plane $y = 0$.

ANALYTICAL TREATMENT

The solution will be sought in the form of a distribution of sources on the surface S of the spheroid. The determination of the surface density σ of this distribution is the object of the following analysis.

Let the x axis coincide with the major axis of the spheroid situated at a depth d below the free surface. Choose the y axis vertical and upwards and let the origin be at the center of the spheroid. Let the stream, of constant velocity U, flow in the positive x direction. The velocity potential corresponding to the three-dimensional motion past a unit source at (ξ, η, ζ) is given by

$$\phi_s = Ux - \frac{1}{R} + G(x,y,z;\xi,\eta,\zeta) \tag{26}$$

where

$$G(x,y,z;\xi,\eta,\zeta) = Re\left[\frac{1}{2\pi}\int_0^{2\pi} PV\int_0^{\infty} \frac{k+k_0\sec^2 t}{k-k_0\sec^2 t}\, e^{-k(d-\eta)}\, e^{k(y-d)+ik\left[(x-\xi)\cos t+(z-\zeta)\sin t\right]}\, dk\, dt\right] + Re\left[2k_0 i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 t\, e^{-k_0\sec^2 t\,(d-\eta)}\, e^{k_0\sec^2 t\left[(y-d)+i(x-\xi)\cos t+i(z-\zeta)\sin t\right]}\, dt\right] \tag{27}$$

$$y < 2d - \eta$$

Here Re denotes the real part of a complex number,* $k_o = g/U^2$, g is the acceleration of gravity, and

$$R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

The harmonic function $G(x,y,z;\xi,\eta,\zeta)$ is sometimes called the Havelock potential since Havelock (1932) was the first one to express it as a double integral in terms of Rayleigh's fictitious viscosity. We have then for the velocity potential Φ corresponding to the distribution of sources on the surface S of the spheroid the expression

$$\Phi = Ux - \int_S \frac{1}{R}\sigma(\xi,\eta,\zeta)\,dS + \int_S \sigma(\xi,\eta,\zeta)\,G(x,y,z;\xi,\eta,\zeta)\,dS \tag{28}$$

$$y < 2d - b$$

where 2b is the length of the minor axis of the spheroid and the integrations over the surface S of the spheroid are performed in the variables ξ, η, and ζ.

Since (26) satisfies the free-surface boundary condition, the boundary condition at $y = -\infty$, and the radiation condition at $x = -\infty$, so does the velocity potential Φ given by (28). The unknown surface density σ must therefore be determined so that the boundary condition the surface of the body,

$$\frac{\partial\Phi}{\partial n} = 0$$

---

*The symbol Re could be left out since the imaginary parts of the two expressions within brackets are identically zero.

is satisfied. When Φ, as given by (28), is substituted into this boundary condition, a Fredholm integral equation of the second kind is obtained for σ,

$$2\pi\sigma(x,y,z)+\int_S \sigma(\xi,\eta,\zeta)\frac{\partial}{\partial n}\left\{-\frac{1}{R}+G(x,y,z;\xi,\eta,\zeta)\right\}dS=-U\frac{\partial x}{\partial n} \tag{29}$$

The solution of this integral equation will be sought by expanding the functions involved in series of spheroidal harmonics, thus taking advantage of the particular shape of the body.

On changing the order in which the integrations are performed, we can write Φ in the equivalent form

$$\begin{aligned}\Phi = {} & Ux-\int_S \frac{1}{R}\sigma(\xi,\eta,\zeta)\,dS+\mathrm{Re}\Bigg[\frac{1}{2\pi}\int_0^{2\pi}PV\!\!\int_0^{\infty}\frac{k+k_0\sec^2 t}{k-k_0\sec^2 t}\,e^{-2kd} \\ & e^{ky+ik(x\cos t+z\sin t)}\left\{\int_S e^{k\eta-ik(\xi\cos t+\zeta\sin t)}\sigma(\xi,\eta,\zeta)\,dS\right\}dk\,dt\Bigg] \\ & +\mathrm{Re}\Bigg[2k_0 i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sec^2 t\,e^{-2k_0 d\sec^2 t}\,e^{k_0\sec^2 t\left[y+i(x\cos t+z\sin t)\right]} \\ & \left\{\int_S e^{k_0\sec^2 t\left[\eta-i(\xi\cos t+\zeta\sin t)\right]}\sigma(\xi,\eta,\zeta)\,dS\right\}dt\Bigg]\end{aligned} \tag{30}$$

We will denote the length of the major axis of the spheroid by $2a$ and its focal distance by c. Let η, θ, and φ be the orthogonal confocal coordinate system defined by

$$\begin{aligned} x &= c\cosh\eta\;\cos\theta\,, \\ y &= c\sinh\eta\;\sin\theta\;\cos\varphi\,, \\ z &= c\sinh\eta\;\sin\theta\;\sin\varphi\,. \end{aligned} \tag{31}$$

Th surfaces η = constant are confocal prolate ellipsoids of revolution, with common foci at the points $(\pm c,0,0)$. The surfaces θ = constant are confocal hyperboloids of revolution, with the same foci. Let η_o be the value of the confocal coordinate η that corresponds to our spheroid. We have

$$a = c \cosh \eta_o, \qquad b = c \sinh \eta_o \tag{32}$$

Let $f(\theta,\varphi)$ be the value of the Newtonian potential

$$\phi_1 = -\int_S \frac{1}{R} \sigma(\xi,\eta,\zeta)\, dS$$

on the surface of the spheroid, and let it be represented by a series of surface spherical harmonics.*

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*Hobson's notation, which is also the one chosen by the National Bureau of Standards in its Handbook of Mathematical Functions, is used in this work. We have

$$P_n^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m P_n(z)}{(dz)^m},$$

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{(dx)^m},$$

$$Q_n^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m Q_n(z)}{(dz)^m},$$

$$Q_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m Q_n(x)}{(dx)^m},$$

where z is any complex number not real and between -1 and 1, x is a real number in the interval $-1 \leq x \leq 1$, n and m are positive integers or zero, $m \leq n$. The only points of discontinuity of the functions $P_n^m(z)$ $(m \neq 0)$ and $Q_n^m(z)$ are on the straight line $(-1,+1)$.

$$f(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_n^m \, P_n^m(\cos\theta) \cos m\varphi \tag{33}$$

We assume this series to be absolutely and uniformly convergent over the surface S. We can then write (Hobson 1931)

$$\phi_1 = -\int_S \frac{1}{R}\sigma(\xi,\eta,\zeta)\, dS = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_n^m \, P_n^m(\cos\theta) \cos m\varphi \, \frac{Q_n^m(\cosh\eta)}{Q_n^m(\cosh\eta_0)} \tag{34}$$

this being the potential function for the space external to the spheroid which converges to $f(\theta,\varphi)$ on the boundary. For the potential function for the internal space which converges to $f(\theta,\varphi)$ over the boundary we have the corresponding series expansion

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} a_n^m \, P_n^m(\cos\theta) \cos m\varphi \, \frac{P_n^m(\cosh\eta)}{P_n^m(\cosh\eta_0)} \tag{35}$$

The surface density σ can be expressed in terms of the coefficients a_n^m of these expansions. We obtain (see Appendix A)

$$\sigma = U \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_n^m \frac{(-1)^{m+1}}{4\pi} \, \frac{P_n^m(\cos\theta)\cos m\varphi}{P_n^m(\cosh\eta_0)\,\sinh\eta_0\,(\cosh^2\eta_0 - \cos^2\theta)^{\frac{1}{2}}} \tag{36}$$

where, for convenience, we have put

$$A_n^m = a_n^m \, \frac{(n+m)!}{(n-m)!} \, \frac{1}{U c \, Q_n^m(\cosh\eta_0)} \tag{37}$$

Substituting this expression for σ in the surface integrals contained in the third and fourth terms on the right side of (30), we find for the first of these

$$\int_S e^{k\eta - ik(\xi \cos t + \zeta \sin t)} \sigma(\xi,\eta,\zeta)\, dS = \frac{Uc^2}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n+1}$$

$$i^{n+m} A_n^m \left\{ (\sec t + \tan t)^m + \frac{1}{(\sec t + \tan t)^m} \right\} \sqrt{\frac{\pi}{2\Delta}}\, J_{n+\frac{1}{2}}(\Delta) \qquad (38)$$

where

$$\Delta = k\, c \cos t,$$

since (see Appendix C, Equation 22 with i replaced by $-i$)

$$\int_S e^{k\eta - ik(\xi \cos t + \zeta \sin t)} P_n^m(\cos\theta) \cos m\varphi \frac{dS}{c^2 \sinh \eta_0 (\cosh^2\eta_0 - \cos^2\theta)^{\frac{1}{2}}}$$

$$= \int_0^{\pi}\int_0^{2\pi} e^{kb \sin\theta \cos\varphi - ik(a \cos\theta \cos t + b \sin\theta \sin\varphi \sin t)} P_n^m(\cos\theta) \cos m\varphi \sin\theta\, d\theta\, d\varphi$$

$$= 2\pi(-1)^{n+m} i^{n+m} P_n^m(\cosh\eta_0) \left\{ (\sec t + \tan t)^m + \frac{1}{(\sec t + \tan t)^m} \right\} \sqrt{\frac{\pi}{2\Delta}}\, J_{n+\frac{1}{2}}(\Delta) \qquad (39)$$

(We assume that the integration can be carried out term by term.) The surface integral contained in the fourth term on the right side of (30) is given by (38) with k replaced by $k_o \sec^2 t$.

Finally, using (38) and the corresponding expression for the surface integral in the fourth term, and expanding the functions

$$e^{ky + ik(x \cos t + z \sin t)}, \qquad e^{k_o \sec^2 t \left[y + i(x \cos t + z \sin t) \right]}$$

in series of spheroidal harmonics, we can express the velocity potential Φ in the form

$$\Phi = Uc\cosh\eta\cos\theta + Uc\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\cos m\varphi\, Q_n^m(\cosh\eta)$$

$$+ R_e\left[\frac{1}{2\pi}\int_0^{2\pi} PV\int_0^{\infty} \frac{k+k_o\sec^2 t}{k-k_o\sec^2 t} e^{-2kd}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{n,m}\cos m\varphi\, P_n^m(\cos\theta)\right.\right.$$

$$\left.\left.P_n^m(\cosh\eta)\right\}\left\{Uc^2\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'} \frac{(-1)^{m'+1}}{2} i^{n'+m'} A_{n'}^{m'} T_{n'}^{m'}(k,t)\sqrt{\frac{2\pi}{kc\cos t}}\right\} dk\,dt\right]$$

$$+ R_e\left[2k_o i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 t\, e^{-2k_o d\sec^2 t}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{o_{n,m}}\cos m\varphi\, P_n^m(\cos\theta)\right.\right.$$

$$\left.\left.P_n^m(\cosh\eta)\right\}\left\{Uc^2\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'} \frac{(-1)^{m'+1}}{2} i^{n'+m'} A_{n'}^{m'} T_{o_{n'}}^{m'}(t)\sqrt{\frac{2\pi}{k_o c\sec t}}\right\} dt\right] \tag{40}$$

where the coefficients $C_{n,m}$ are given by (24) of Appendix C, with $C_{n,o} = C_n$, the coefficients $B_{n,m}$ have been omitted because the potential is an even function of z, the coefficients $C_{o_{n,m}}$ are obtained from the coefficients $C_{n,m}$ by putting $k = k_o \sec^2 t$, and, for brevity, the functions

$$T_n^m(k,t) = \frac{1}{2}\left\{(\sec t+\tan t)^m + \frac{1}{(\sec t+\tan t)^m}\right\} J_{n+\frac{1}{2}}(kc\cos t) \tag{41}$$

and

$$T_{o_n}^m(t) = T_n^m(k_o\sec^2 t, t)$$

have been introduced. Again assuming that integration of the series term by term is possible, and replacing the coefficients $C_{n,m}$ and $C_{o_{n,m}}$ by their expressions in terms of the functions T_n^m and $T_{o_n}^m$,

$$C_{n,m} = i^{n+m}\frac{(n-m)!}{(n+m)!}(2n+1)\sqrt{\frac{2\pi}{kc\cos t}}\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix} T_n^m(k,t)$$

$$C_{o_{n,m}} = i^{n+m}\frac{(n-m)!}{(n+m)!}(2n+1)\sqrt{\frac{2\pi}{k_o c\sec t}}\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix} T_{o_n}^m(t)$$

we can write (40) in the equivalent form

$$\Phi = Uc\cosh\eta\cos\theta + Uc\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\cos m\varphi\, Q_n^m(\cosh\eta)$$

$$+ Uc\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\cos m\varphi\, P_n^m(\cosh\eta)\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix}\frac{1}{2}\frac{(n-m)!}{(n+m)!}(2n+1)\, R_e\Big[i^{n+m}$$

$$\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}(-1)^{n'+1} i^{n'+m'} A_{n'}^{m'} \int_0^{2\pi} PV\int_0^{\infty}\frac{k+k_0\sec^2 t}{k-k_0\sec^2 t}\frac{e^{-2kd}}{k\cos t} T_n^m(k,t)\, T_{n'}^{m'}(k,t)\,dk\,dt\Big]$$

$$+ Uc\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\cos m\varphi\, P_n^m(\cosh\eta)\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix}\frac{1}{2}\frac{(n-m)!}{(n+m)!}(2n+1)\, Re\Big[i^{n+m}$$

$$4\pi i\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}(-1)^{n'+1} i^{n'+m'} A_{n'}^{m'}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-2k_0 d\sec^2 t}}{\cos t} T_{o_n}^{m}(t)\, T_{o_{n'}}^{m'}(t)\,dt\Big] \qquad (42)$$

where the factor 1/2 is to be used for m = 0. Substitution of this expression for the velocity potential into the boundary condition on the surface of the spheroid,

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{or} \quad \frac{\partial\Phi}{\partial\eta} = 0$$

yields

$$\cos\theta + \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\cos m\varphi\, A_n^m\frac{(n-m)!}{(n+m)!}\dot{Q}_n^m(\cosh\eta_0) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}$$

$$P_n^m(\cos\theta)\cos m\varphi\, \dot{P}_n^m(\cosh\eta_0)\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix}\frac{1}{2}\frac{(n-m)!}{(n+m)!}(2n+1)\, R_e\Big[i^{n+m}$$

$$\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}(-1)^{n'+1} i^{n'+m'} A_{n'}^{m'}\int_0^{2\pi} PV\int_0^{\infty}\frac{k+k_0\sec^2 t}{k-k_0\sec^2 t}\frac{e^{-2kd}}{k\cos t} T_n^m(k,t)\, T_{n'}^{m'}(k,t)\,dk\,dt\Big]$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\cos m\varphi\, \dot{P}_n^m(\cosh\eta_0)\begin{Bmatrix}1\\ \frac{1}{2}\end{Bmatrix}\frac{1}{2}\frac{(n-m)!}{(n+m)!}(2n+1)\, Re\Big[i^{n+m}\, 4\pi i$$

$$\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}(-1)^{n'+1} i^{n'+m'} A_{n'}^{m'}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-2k_0 d\sec^2 t}}{\cos t} T_{o_n}^{m}(t)\, T_{o_{n'}}^{m'}(t)\,dt\Big] = 0 \qquad (43)$$

where we have now assumed that the series can be differentiated term by term; the dot above the Legendre functions P and Q indicates differentiation with respect to the argument $\cosh \eta$. Equation (43) is equivalent to the infinite set of equations

$$A_n^m = \Gamma_n^m + \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} {}_n^m B_{m'}^{n'} A_{m'}^{n'} \tag{44}$$

where

$$\Gamma_n^m = \begin{cases} -\dfrac{1}{\dot{Q}_1(\cosh \eta_0)} & \text{if } n = 1,\ m = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$${}_n^m B_{m'}^{n'} = \frac{\dot{P}_n^m(\cosh \eta_0)}{\dot{Q}_n^m(\cosh \eta_0)} \left\{ \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right\} \frac{1}{2}(2n+1)(-1)^{n'} \, {}_n^m I_{m'}^{n'}$$

with

$${}_n^m I_{m'}^{n'} = \begin{cases} (-1)^{\frac{n+m+n'+m'}{2}} 4 \displaystyle\int_0^{\frac{\pi}{2}} PV \int_0^{\infty} \frac{k+k_0 \sec^2 t}{k-k_0 \sec^2 t} \frac{e^{-2kd}}{k \cos t} T_n^m(k,t) T_{n'}^{m'}(k,t)\, dk\, dt & (n+m+n'+m') \text{ even} \\ (-1)^{\frac{n+m+n'+m'+1}{2}} 8\pi \displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{-2k_0 d \sec^2 t}}{\cos t} T_{o_n}^m(t) T_{o_{n'}}^{m'}(t)\, dt & (n+m+n'+m') \text{ odd} \end{cases}$$

CALCULATION OF THE WAVE RESISTANCE

Lagally's theorem (Lagally 1922) yields for the horizontal component of the force on the spheroid the expression

$$R = -4\pi\rho \int_S \sigma \frac{\partial \phi_2}{\partial x} \, dS \tag{45}$$

where ϕ_2 is the Havelock potential for the source distribution on the surface of the spheroid, given by the third and fourth terms on the right side of Equation (30). We have

$$\int_S \sigma \frac{\partial \phi_2}{\partial x} \, dS = Re \Bigg[\frac{1}{2\pi} \int_0^{2\pi} PV \int_0^{\infty} \frac{k + k_0 \sec^2 t}{k - k_0 \sec^2 t} \, e^{-2kd} \, ik \cos t$$

$$\left\{ \int_S e^{ky + ik(x\cos t + z\sin t)} \sigma(x,y,z) \, dS \right\} \left\{ \int_{S'} e^{k\eta - ik(\xi\cos t + \zeta\sin t)} \sigma(\xi,\eta,\zeta) \, dS' \right\} dk \, dt \Bigg]$$

$$+ Re \Bigg[2k_0 i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 t \, e^{-2k_0 d \sec^2 t} \, i k_0 \sec t \left\{ \int_S e^{k_0 \sec^2 t \left[y + i(x\cos t + z\sin t) \right]} \sigma(x,y,z) \, dS \right\}$$

$$\left\{ \int_{S'} e^{k_0 \sec^2 t \left[\eta - i(\xi\cos t + \zeta\sin t) \right]} \sigma(\xi,\eta,\zeta) \, dS' \right\} dt \Bigg] \tag{46}$$

where both surface integrations are performed over the surface of the spheroid, in the variables x, y, and z when we denote it by S, and

in the variables ξ, η, and ζ when we denote it by S'. The contribution to the wave resistance from the first term on the right side of (46) is zero. To see this we need only write

$$Re\left[i \left\{ \int_S e^{ky+ik(x\cos t+z\sin t)} \sigma(x,y,z)\,dS \right\} \left\{ \int_{S'} e^{k\eta-ik(\xi\cos t+\zeta\sin t)} \sigma(\xi,\eta,\zeta)\,dS' \right\} \right]$$

$$= -\int_S \int_{S'} e^{k(y+\eta)} \sin\left[k(x-\xi)\cos t + k(z-\zeta)\sin t \right] \sigma(\xi,\eta,\zeta)\,\sigma(x,y,z)\,dS\,dS'$$

and observe that if we interchange x, y, z, with ξ, η, ζ, the integrand changes sign while its absolute value remains unchanged. To evaluate the contribution to the wave resistance from the second term on the right side of (46), we make use of (38) with k replaced by $k_o \sec^2 t$. We obtain

$$\int_S \sigma \frac{\partial \phi_2}{\partial x}\,dS = Re\left[2k_o i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 t\, e^{-2k_o d \sec^2 t} \, i k_o \sec t \left(\frac{Uc^2}{2} \right)^2 \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{m+1} i^{n+m} A_n^m \sqrt{\frac{2\pi}{k_o c \sec t}}\, T_{o_n}^m(t) \right\} \left\{ \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} (-1)^{m'+1} i^{n'+m'} A_{n'}^{m'} \sqrt{\frac{2\pi}{k_o c \sec t}}\, T_{o_{n'}}^{m'}(t) \right\} dt \right]$$

The wave resistance is then given by

$$R = \pi \rho g c^3 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} (-1)^{m+m'} A_n^m A_{n'}^{m'} \, {}_n^m I_{R\,n'}^{m'} \tag{47}$$

(n+m+n'+m') even

where

$$ {}_{n}^{m}I_{K_{n'}}^{m'} = (-1)^{\frac{n \pm m + n' \mp m'}{2}} 8\pi \int_{0}^{\frac{\pi}{2}} \sec^2 t \, e^{-2k_0 d \sec^2 t} \, T_{o_{n}}^{m}(t) \, T_{o_{n'}}^{m'}(t) \, dt $$

NUMERICAL EVALUATIONS

All numerical evaluations were performed with an IBM 360/65 computer. The corresponding programs can be found in Appendix E and will be discussed in this section. Subprograms which are common to various programs appear only once in the Appendix.

In order to evaluate the single integrals in the coefficients of the infinite system of equations (44), we introduce the change of variable

$$\tan t = u.$$

Let

$$I_1 = 8\pi \int_0^{\frac{\pi}{2}} \frac{e^{-2k_0 d \sec^2 t}}{\cos t}\, T_{0_m}^{m}(t)\, T_{0_{m'}}^{m'}(t)\, dt \tag{48}$$

We have

$$I_1 = 8\pi\, e^{-2k_0 d} \int_0^{\infty} \frac{e^{-2k_0 d u^2}}{\sqrt{1+u^2}}\, T_{ou_m}^{m}(u)\, T_{ou_{m'}}^{m'}(u)\, du \tag{49}$$

where

$$T_{ou_m}^{m}(u) = \frac{1}{2}\left\{\left(\sqrt{1+u^2}+u\right)^m + \frac{1}{\left(\sqrt{1+u^2}+u\right)^m}\right\} J_{m+\frac{1}{2}}\left(k_0 c\sqrt{1+u^2}\right)$$

Since the integrand in (49) is an even function of u, we can write

$$I_1 = 8\pi \frac{e^{-2k_0 d}}{\sqrt{2k_0 d}}\, \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, \frac{T_{ou_m}^{m}\left(\frac{x}{\sqrt{2k_0 d}}\right)\, T_{ou_{m'}}^{m'}\left(\frac{x}{\sqrt{2k_0 d}}\right)}{\sqrt{1+\left(\frac{x}{\sqrt{2k_0 d}}\right)^2}}\, dx \tag{50}$$

where we have further changed the variable of integration to

$$x = \sqrt{2k_o d}\ u$$

The integral in (50) can be evaluated by Hermite-Gauss quadrature. A twenty-point formula was used, for which the zeros and weighting factors can be found in Abramowitz and Stegun (1964). Actually, since the integrand is an even function of x, only ten ordinates are evaluated in the corresponding computer program, which yields the value of I_1 and includes a function subprogram (BESSJ) to evaluate the Bessel functions which appear in the integrand.

The evaluation of the Bessel functions J of order $n + 1/2$ was carried out using the series expression

$$J_{n+\frac{1}{2}}(z) = \frac{\left(\frac{z}{2}\right)^{n+\frac{1}{2}}}{\Gamma\left(n+\frac{3}{2}\right)} \left\{ 1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2.4(2n+3)(2n+5)} - \cdots \right\} \tag{51}$$

for values of the argument less than five, and the recurrence relation

$$J_{n+\frac{1}{2}}(z) = \frac{2n-1}{z} J_{n-\frac{1}{2}} - J_{n-\frac{3}{2}} \tag{52}$$

for values greater than five. In this manner, the large round-off errors associated with the use of (51) for large values of the argument (especially for small n), and with the use of (52) for small values of the argument, are avoided. The cut-off value of five is a rough estimate of the value necessary to minimize these errors. The subprogram, which was written in double precision further to reduce round-off errors (only six, or at most seven, significant digits are supplied by the IBM 360/65 computer in single precision), was checked against

tables included in Watson's treatise (Watson 1944, pp. 740-743). The results were found to be accurate to the number of significant figures given in these tables (six or less) for values of n up to 18; most of them are probably accurate to several more significant figures. The round-off errors associated with (52), however, increase with increasing n, and this high accuracy should not be expected to obtain in the evaluation of Bessel functions of much larger order.

In view of the difficulty of estimating the error in the Hermite-Gauss quadrature formula, the calculations were spot-checked using Simpson's rule, which requires longer computing times, but allows for a ready estimate of the error at each stage of approximation. The subroutine SMPSN used in the corresponding program is part of the IBM System/360 Scientific Subroutine Package, Version II, and is not included in Appendix E. In this subroutine, Simpson's rule is used with interval halving until the difference between successive values of the integral is less than a given tolerance. The Hermite-Gauss quadrature formula was found to provide at least five significant figures.

To evaluate the double integrals in the coefficients of (44), we make use again of the change of variable

$$\tan t = u$$

and we put further

$$k = k_o K$$

Let

$$I_2 = 4\int_0^{\frac{\pi}{2}} PV\int_0^{\infty} \frac{k+k_o\sec^2 t}{k-k_o\sec^2 t}\,\frac{e^{-2kd}}{k\cos t}\,T_m(k,t)\,T_{m'}(k,t)\,dk\,dt \quad (53)$$

We have then

$$I_z = \int_0^\infty PV \int_0^\infty \frac{K+(1+u^2)}{K-(1+u^2)} \frac{e^{-2k_0 dK}}{K\sqrt{1+u^2}} \left\{ \left(\sqrt{1+u^2}+u\right)^m + \frac{1}{\left(\sqrt{1+u^2}+u\right)^m} \right\}$$

$$\left\{ \left(\sqrt{1+u^2}+u\right)^{m'} + \frac{1}{\left(\sqrt{1+u^2}+u\right)^{m'}} \right\} J_{m+\frac{1}{2}}\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right) J_{m'+\frac{1}{2}}\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right) dK\, du \tag{54}$$

For numerical evaluation, it is better to write (54) in the form

$$I_z = k_0 c \int_0^\infty \left\{ \left(1+\frac{u}{\sqrt{1+u^2}}\right)^m + \frac{1}{\left(1+\frac{u}{\sqrt{1+u^2}}\right)^m (1+u^2)^m} \right\} \left(\sqrt{1+u^2}\right)^{-(m+m'+2-m-m')}$$

$$\left\{ \left(1+\frac{u}{\sqrt{1+u^2}}\right)^{m'} + \frac{1}{\left(1+\frac{u}{\sqrt{1+u^2}}\right)^{m'} (1+u^2)^{m'}} \right\} \left\{ PV \int_0^\infty \frac{K+(1+u^2)}{K-(1+u^2)} e^{-2k_0 dK} \right.$$

$$\left. (k_0 cK)^{m+m'} \frac{J_{m+\frac{1}{2}}\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right)}{\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right)^{m+\frac{1}{2}}} \frac{J_{m'+\frac{1}{2}}\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right)}{\left(\frac{k_0 cK}{\sqrt{1+u^2}}\right)^{m'+\frac{1}{2}}} dK \right\} du = k_0 c \int_0^\infty G(u)\, du \tag{55}$$

where the function

$$\frac{J_{m+\frac{1}{2}}(z)}{z^{m+\frac{1}{2}}} = \frac{1}{2^{m+\frac{1}{2}}\,\Gamma\left(m+\frac{3}{2}\right)} \left\{ 1 - \frac{z^2}{2(2m+3)} + \ldots \right\} \tag{56}$$

is continuous at $z = 0$. A simple modification of the function subprogram used to evaluate the Bessel functions yields the function subprogram (BESSJM) used to evaluate (56).

The evaluation of the Cauchy principal-value integral in (55) was

carried out using Simpson's rule, modified according to a technique suggested by L. Landweber (Kobus 1967, Appendix 1) to take into account the singular point in the integrand. It can be shown that any composite integration rule, obtained by dividing the interval of integration into subranges and applying in each subrange any of the Newton-Cotes integration formulae, can be used to evaluate a Cauchy principal-value integral,

$$\int_a^b \frac{f(x)}{x-c}\,dx, \qquad a < c < b$$

as long as the singular point c is either an end point (excluding a and b) or the middle point of a subrange, and the value of the integrand at c is taken equal to the derivative $f'(c)$. In the function subprogram that evaluates the function $G(u)$ defined in (55), therefore, the derivative at the singular point $k = 1+u^2$ is obtained first using a five-point differentiation formula. The two function subprograms FM and F evaluate the integrand of the principal-value integral and, at the singular point, replace the integrand with the corresponding derivative. The calculations were carried out to an accuracy of three significant figures.

The solution of the infinite system of equations (44) was obtained by using the so-called method of reduction (Kantorovich and Krylov 1958, pp. 25-26, 30-31). In this method, the solution is found by solving a sequence of finite systems, each of which is obtained from the infinite system by discarding all equations and unknowns beyond a certain number N. The solutions of this sequence of systems, under certain conditions, converge to the (principal) solution of the infinite system. Although

it is difficult in our case to show that the conditions of the theorem are satisfied, it is reasonable to assume, on account of the significance of (44), that the method does yield the solution sought. A numerical verification of this assumption is obtained if, when the number N of equations kept is increased, the coefficients A_n^m approach limiting values.

Such a numerical verification has been included in the computer program for the solution of (44). The number of equations has been taken equal to 2, 5, 9, 14, and 20, equivalent to keeping, in expansions (34) and (36), only those terms containing Legendre functions of degrees up to 1, 2, 3, 4, and 5, respectively. The program contains a subroutine subprogram for computing the coefficient matrix ${}_n^m B_{n'}^{m'}$ from the symetric matrix formed by the double and single integrals ${}_n^m I_{n'}^{m'}$. No table was found to give the values of the Legendre functions and their derivatives in the interval of interest to our work, and their evaluation is carried out in a second subprogram. The derivatives occur in the coefficients ${}_n^m B_{n'}^{m'}$; the Legendre functions of the first kind in expansion (36). No use is made in this second subprogram of the recurrence relations for the Legendre functions, on which, for example, the tables published by the National Bureau of Standards (Lowan 1945) are based; severe round-off errors were found to be associated with the use of these relations, in particular near $x = 1$. Instead, since only the functions of degrees up to five were needed, the explicit expressions of these functions were derived and used in the computations. These expressions were checked, using the National Bureau of Standard tables, for several values of the argument; all significant figures

given in the tables were found to be correct. The subroutine GELG used in the program for the solution of (44) is part of the IBM System/360 Scientific Subroutine Package, Version II, and is not included in Appendix E. In this subroutine, the solution of a system of general simultaneous linear equations is obtained by means of Gauss elimination with complete pivoting.

Once the coefficients A_n^m are known, we can evaluate the density of the source distribution making use of (36). A program was prepared to obtain plots of this density along meridian curves of the spheroid. The subroutine PNMXG1 used in this program, which evaluates the Legendre associated functions of the first kind for values of the argument greater than one, is simply part of the subroutine LEGF; the subroutine PNMXL1, which computes these functions for values of the argument less than one, was obtained by introducing the necessary changes in sign in the expressions in the subroutine PNMXG1. Neither subroutine is included in Appendix E.

The evaluation of the wave resistance is carried out in the last program in Appendix E. A simple modification of the program for the evaluation of the single integral in the coefficients of (44) yields the program for evaluating the single integrals in the expression (47) for the wave resistance; the modified program is not included.

RESULTS AND DISCUSSION

The double and single integrals ${}^{m}_{n}I^{m'}_{n'}$ in the coefficients of system (44) depend on the Froude number $U/\sqrt{2gc} = 1/\sqrt{2k_o c}$, and on the relative depth of submergence d/c, and are independent of the eccentricity c/a of the spheroid, which occurs in the derivatives of the Legendre functions. This property of the coefficients of system (44) is of interest because most of the computer time required in order to evaluate the coefficients A^m_n is spent in the evaluation of the double integrals; once these integrals have been computed, for given values of the Froude number and the relative depth of submergence, the coefficients A^m_n can be obtained for several values of the eccentricity without practically increasing the total computing time. The numerical evaluations reported here were carried out for a Froude number $U/\sqrt{2gc} = 0.4$, a relative depth of submergence $d/c = 0.5$, and reciprocal eccentricities $a/c = 1.01, 1.02, 1.04, 1.06, 1.08,$ and 1.10, corresponding to slenderness ratios $a/b = 7.12, 5.07, 3.64, 3.02, 2.65,$ and 2.40, respectively. (For the relative depth of submergence chosen, the spheroid pierces the free surface for $a/c = \sqrt{1.25}$.) The practical interest of the smaller slenderness ratios is probably limited; they are included here for purposes of comparison and discussion of results.

The first approximation to the solution of system (44) which consists of neglecting entirely the infinite series on the right side, thus

keeping only the first term of expansion (34) with

$$\frac{a_1^0}{U c\, Q_1\left(\frac{a}{c}\right)} = A_1^0 = \frac{1}{\dot{Q}_1\left(\frac{a}{c}\right)} = \frac{1}{\frac{a/c}{(a/c)^2-1} - \frac{1}{2}\ln\frac{a/c+1}{a/c-1}} \tag{57}$$

corresponds to the source distribution that produces the spheroid in an unbounded uniform stream without a free surface. Indeed, the potential for the motion of a prolate spheroid parallel to its axis (in the negative x direction) in an infinite mass of liquid is given by

$$\phi = U c \frac{1}{\dot{Q}_1\left(\frac{a}{c}\right)} P_1(\cos\theta)\, Q_1(\cosh\eta) \tag{58}$$

or

$$\phi = \frac{U c}{\frac{\zeta_0}{\zeta_0^2-1} - \frac{1}{2}\ln\frac{\zeta_0+1}{\zeta_0-1}}\, \mu \left(\frac{1}{2}\zeta \ln\frac{\zeta+1}{\zeta-1} - 1\right)$$

with $\zeta_0 = \cosh\eta_0 = a/c$, $\zeta = \cosh\eta$, $\mu = \cos\theta$. For this first approximation, the wave resistance is given by

$$R = \pi\rho g c^2 \left(A_1^0\right)^2 8\pi \int_0^{\frac{\pi}{2}} \sec^2 t\, e^{-2k_0 d \sec^2 t} \left\{ J_{\frac{3}{2}}\left(\frac{k_0 c}{\cos t}\right)\right\}^2 dt \tag{59}$$

Equation (59) was obtained by Havelock (1931a) using the axial source distribution corresponding to the motion of the spheroid in an infinite mass of liquid and applying Lagally's theorem to evaluate the wave resistance (without, however, mentioning the name of Lagally in this

connection). Of course, neither the axial source distribution nor the source distribution on the surface of the spheroid actually produce a spheroid in the presence of a free surface. They are two different first approximations. It is interesting to note, however, that both first approximations yield the same value as a first approximation for the wave resistance. Havelock's approximation has been included in Tables 1 through 5 for purposes of comparison.

The values of the coefficients A_n^m, obtained keeping 1, 2, 5, 9, 14, and 20 equations (and an equal number of coefficients) in (44), are presented in Tables 1 through 4, for four values of the ratio a/c. Table 5 contains the corresponding successive approximations to the wave resistance (the two additional a/c ratios investigated are also included in this table). Three significant figures are given for all elements in the tables, since the double integrals were computed to this accuracy. When 20 equations are kept, 92 double integrals must be evaluated (the integrals ${}^{m}_{n}I^{m'}_{n'}$ satisfy the symmetry relations ${}^{m}_{n}I^{m'}_{n'} = {}^{m'}_{n'}I^{m}_{n}$ and ${}^{m}_{n}I^{m'}_{n'} = {}^{m'}_{n}I^{m}_{n'}$, and this reduces the number of double integrals to be evaluated from 202 to 92); the required computing time is a little less than two hours. The corresponding computing time for the evaluation of the single integrals is less than half a minute; for the solution of (44), for the six ratios a/c investigated, including the evaluation of the derivatives of the Legendre functions which appear in the coefficients of the system, less than one minute. Also less than one minute is the computing time required for the evaluation of the wave resistance, using (47); only single integrals appear in this expression. A reduction in the time required for the evaluation of each double integral, at

Table 1. Coefficients A_n^m for $a/c = 1.01$

No. of coeffs. kept						
1	$A_1^0 = 0.212\times10^{-1}$			Havelock's approximation: $A_1^0 = 0.210\times10^{-1}$		
2	$A_1^0 = 0.212\times10^{-1}$	$A_1^1 = -0.573\times10^{-4}$				
5	$A_1^0 = 0.212\times10^{-1}$	$A_1^1 = -0.611\times10^{-4}$				
	$A_2^0 = 0.173\times10^{-3}$	$A_2^1 = 0.789\times10^{-3}$	$A_2^2 = -0.380\times10^{-5}$			
9	$A_1^0 = 0.212\times10^{-1}$	$A_1^1 = -0.635\times10^{-4}$				
	$A_2^0 = 0.176\times10^{-3}$	$A_2^1 = 0.786\times10^{-3}$	$A_2^2 = -0.393\times10^{-5}$			
	$A_3^0 = -0.$[illegible]7×10^{-3}	$A_3^1 = 0.546\times10^{-3}$	$A_3^2 = 0.304\times10^{-4}$	$A_3^3 = -0.107\times10^{-6}$		
14	$A_1^0 = 0.212\times10^{-1}$	$A_1^1 = -0.641\times10^{-4}$				
	$A_2^0 = 0.178\times10^{-1}$	$A_2^1 = 0.785\times10^{-3}$	$A_2^2 = -0.397\times10^{-5}$			
	$A_3^0 = -0.385\times10^{-1}$	$A_3^1 = 0.549\times10^{-3}$	$A_3^2 = 0.303\times10^{-4}$	$A_3^3 = -0.109\times10^{-6}$		
	$A_4^0 = -0.306\times10^{-3}$	$A_4^1 = -0.456\times10^{-3}$	$A_4^2 = 0.275\times10^{-4}$	$A_3^3 = 0.791\times10^{-}$[illegible]	$A_4^4 = -0.208\times10^{-8}$	
20	$A_1^0 = 0.212\times10^{-1}$	$A_1^1 = -0.642\times10^{-4}$				
	$A_2^0 = 0.178\times10^{-3}$	$A_2^1 = 0.785\times10^{-3}$	$A_2^2 = -0.398\times10^{-5}$			
	$A_3^0 = -0.384\times10^{-3}$	$A_3^1 = 0.549\times10^{-3}$	$A_3^2 = 0.302\times10^{-4}$	$A_3^3 = -0.109\times10^{-6}$		
	$A_4^0 = -0.306\times10^{-3}$	$A_4^1 = -0.454\times10^{-3}$	$A_4^2 = 0.275\times10^{-4}$	$A_4^3 = 0.789\times10^{-6}$	$A_4^4 = -0.208\times10^{-8}$	
	$A_5^0 = 0.111\times10^{-3}$	$A_5^1 = -0.331\times10^{-3}$	$A_5^2 = -0.232\times10^{-4}$	$A_5^3 = 0.705\times10^{-6}$	$A_5^4 = 0.183\times10^{-7}$	$A^5 = -0.314\times10^{-10}$

Table 2. Coefficients A_n^m for $a/c = 1.02$

Havelock's approximation: $A_1^0 = 0.436\text{x}10^{-1}$

No. of coeffs. kept						
1	$A_1^0 = 0.443\text{x}10^{-1}$					
2	$A_1^0 = 0.443\text{x}10^{-1}$	$A_1^1 = -0.245\text{x}10^{-3}$				
5	$A_1^0 = 0.443\text{x}10^{-1}$	$A_1^1 = -0.281\text{x}10^{-3}$				
	$A_2^0 = 0.840\text{x}10^{-3}$	$A_2^1 = 0.344\text{x}10^{-2}$	$A_2^2 = -0.352\text{x}10^{-4}$			
9	$A_1^0 = 0.443\text{x}10^{-1}$	$A_1^1 = -0.306\text{x}10^{-3}$				
	$A_2^0 = 0.88$[illegible]$\text{x}10^{-3}$	$A_2^1 = 0.341\text{x}10^{-2}$	$A_2^2 = -0.379\text{x}10^{-4}$			
	$A_3^0 = -0.185\text{x}10^{-2}$	$A_3^1 = 0.268\text{x}10^{-2}$	$A_3^2 = 0.266\text{x}10^{-3}$	$A_3^3 = -0.213\text{x}10^{-5}$		
14	$A_1^0 = 0.443\text{x}10^{-1}$	$A_1^1 = -0.313\text{x}10^{-3}$				
	$A_2^0 = 0.897\text{x}10^{-3}$	$A_2^1 = 0.339\text{x}10^{-2}$	$A_2^2 = -0.387\text{x}10^{-4}$			
	$A_3^0 = -0.183\text{x}10^{-2}$	$A_3^1 = 0.271\text{x}10^{-2}$	$A_3^2 = 0.263\text{x}10^{-3}$	$A_3^3 = -0.219\text{x}10^{-5}$		
	$A_4^0 = -0.171\text{x}10^{-2}$	$A_4^1 = -0.208\text{x}10^{-2}$	$A_4^2 = 0.273\text{x}10^{-3}$	$A_4^3 = 0.138\text{x}10^{-4}$	$A_4^4 = -0.901\text{x}10^{-7}$	
20	$A_1^0 = 0.443\text{x}10^{-1}$	$A_1^1 = -0.314\text{x}10^{-3}$				
	$A_2^0 = 0.898\text{x}10^{-3}$	$A_2^1 = 0.339\text{x}10^{-2}$	$A_2^2 = -0.338\text{x}10^{-4}$			
	$A_3^0 = -0.182\text{x}10^{-2}$	$A_3^1 = 0.271\text{x}10^{-2}$	$A_3^2 = 0.262\text{x}10^{-3}$	$A_3^3 = -0.220\text{x}10^{-5}$		
	$A_4^0 = -0.170\text{x}10^{-2}$	$A_4^1 = -0.206\text{x}10^{-2}$	$A_4^2 = 0.272\text{x}10^{-3}$	$A_4^3 = 0.138\text{x}10^{-4}$	$A_4^4 = -0.907\text{x}10^{-7}$	
	$A_5^0 = 0.522\text{x}10^{-3}$	$A_5^1 = -0.185\text{x}10^{-2}$	$A_5^2 = -0.197\text{x}10^{-3}$	$A_5^3 = 0.146\text{x}10^{-4}$	$A_5^4 = 0.646\text{x}10^{-6}$	$A_5^5 = -0.317\text{x}10^{-8}$

Table 3. Coefficients A_n^m for $a/c = 1.06$

Havelock's approximation: $A_1^0 = 0.147$

No. of coeffs. kept						
1	$A_1^0 = 0.155$					
2	$A_1^0 = 0.155$	$A_1^1 = -0.294\times10^{-2}$				
5	$A_1^0 = 0.156$	$A_1^1 = -0.505\times10^{-2}$				
	$A_2^0 = 0.161\times10^{-1}$	$A_2^1 = 0.451\times10^{-1}$	$A_2^2 = -0.186\times10^{-2}$			
9	$A_1^0 = 0.156$	$A_1^1 = -0.701\times10^{-2}$				
	$A_2^0 = 0.196\times10^{-1}$	$A_2^1 = 0.409\times10^{-1}$	$A_2^2 = -0.250\times10^{-2}$			
	$A_3^0 = -0.239\times10^{-1}$	$A_3^1 = 0.602\times10^{-1}$	$A_3^2 = 0.907\times10^{-2}$	$A_3^3 = -0.472\times10^{-3}$		
14	$A_1^0 = 0.156$	$A_1^1 = -0.740\times10^{-2}$				
	$A_2^0 = 0.199\times10^{-1}$	$A_2^1 = 0.382\times10^{-1}$	$A_2^2 = -0.261\times10^{-2}$			
	$A_3^0 = -0.201\times10^{-1}$	$A_3^1 = 0.598\times10^{-1}$	$A_3^2 = 0.805\times10^{-2}$	$A_3^3 = -0.502\times10^{-3}$		
	$A_4^0 = -0.439\times10^{-1}$	$A_4^1 = -0.136\times10^{-1}$	$A_4^2 = 0.174\times10^{-1}$	$A_4^3 = 0.120\times10^{-2}$	$A_4^4 = -0.791\times10^{-4}$	
20	$A_1^0 = 0.156$	$A_1^1 = -0.739\times10^{-2}$				
	$A_2^0 = 0.197\times10^{-1}$	$A_2^1 = 0.376\times10^{-1}$	$A_2^2 = -0.260\times10^{-2}$			
	$A_3^0 = -0.194\times10^{-1}$	$A_3^1 = 0.585\times10^{-1}$	$A_3^2 = 0.780\times10^{-2}$	$A_3^3 = -0.500\times10^{-3}$		
	$A_4^0 = -0.421\times10^{-1}$	$A_4^1 = -0.120\times10^{-1}$	$A_4^2 = 0.169\times10^{-1}$	$A_4^3 = 0.112\times10^{-2}$	$A_4^4 = -0.792\times10^{-4}$	
	$A_5^0 = -0.726\times10^{-2}$	$A_5^1 = -0.497\times10^{-1}$	$A_5^2 = 0.114\times10^{-3}$	$A_5^3 = 0.307\times10^{-2}$	$A_5^4 = -0.151\times10^{-3}$	$A_5^5 = -0.119\times10^{-4}$

Table 4. Coefficients A_n^m for a/c = 1.10

No. of coeffs. kept						
1	$A_1^0 = 0.298$					
2	$A_1^0 = 0.298$	$A_1^1 = -0.112\times10^{-1}$				
5	$A_1^0 = 0.304$	$A_1^1 = -0.360\times10^{-1}$				
	$A_2^0 = 0.104$	$A_2^1 = 0.171$	$A_2^2 = -0.210\times10^{-1}$			
9	$A_1^0 = 0.294$	$A_1^1 = -0.526\times10^{-1}$				
	$A_2^0 = 0.117$	$A_2^1 = 0.752\times10^{-1}$	$A_2^2 = -0.292\times10^{-1}$			
	$A_3^0 = 0.234\times10^{-1}$	$A_3^1 = 0.344$	$A_3^2 = 0.111\times10^{-1}$	$A_3^3 = -0.101\times10^{-1}$		
14	$A_1^0 = 0.292$	$A_1^1 = -0.477\times10^{-1}$				
	$A_2^0 = 0.990\times10^{-1}$	$A_2^1 = 0.618\times10^{-1}$	$A_2^2 = -0.263\times10^{-1}$			
	$A_3^0 = 0.292\times10^{-1}$	$A_3^1 = 0.270$	$A_3^2 = 0.274\times10^{-2}$	$A_3^3 = -0.928\times10^{-2}$		
	$A_4^0 = -0.147$	$A_4^1 = 0.194$	$A_4^2 = 0.114$	$A_4^3 = -0.523\times10^{-2}$	$A_4^4 = -0.289\times10^{-2}$	
20	$A_1^0 = 0.292$	$A_1^1 = -0.465\times10^{-1}$				
	$A_2^0 = 0.954\times10^{-1}$	$A_2^1 = 0.656\times10^{-1}$	$A_2^2 = -0.256\times10^{-1}$			
	$A_3^0 = 0.194\times10^{-1}$	$A_3^1 = 0.255$	$A_3^2 = 0.423\times10^{-2}$	$A_3^3 = -0.905\times10^{-2}$		
	$A_4^0 = -0.131$	$A_4^1 = 0.166$	$A_4^2 = 0.103$	$A_4^3 = 0.543\times10^{-2}$	$A_4^4 = -0.284\times10^{-2}$	
	$A_5^0 = -0.159$	$A_5^1 = -0.632\times10^{-1}$	$A_5^2 = 0.122$	$A_5^3 = 0.300\times10^{-1}$	$A_5^4 = -0.330\times10^{-2}$	$A_5^5 = -0.867\times10^{-3}$

Table 5. Wave resistance $R/\pi\rho g c^3$

Number of coefficients kept / $\frac{a}{c}$	1.01	1.02	1.04	1.06	1.08	1.10
1	0.401×10^{-4}	0.176×10^{-3}	0.824×10^{-3}	0.215×10^{-2}	0.442×10^{-2}	0.796×10^{-2}
2	0.401×10^{-4}	0.176×10^{-3}	0.824×10^{-3}	0.215×10^{-2}	0.442×10^{-2}	0.796×10^{-2}
5	0.435×10^{-4}	0.208×10^{-3}	0.119×10^{-2}	0.390×10^{-2}	0.104×10^{-1}	0.252×10^{-1}
9	0.446×10^{-4}	0.220×10^{-3}	0.138×10^{-2}	0.507×10^{-2}	0.147×10^{-1}	0.341×10^{-1}
14	0.451×10^{-4}	0.226×10^{-3}	0.145×10^{-2}	0.534×10^{-2}	0.147×10^{-1}	0.310×10^{-1}
20	0.452×10^{-4}	0.227×10^{-3}	0.146×10^{-2}	0.534×10^{-2}	0.144×10^{-1}	0.303×10^{-1}
Havelock's approx.	0.395×10^{-4}	0.170×10^{-3}	0.770×10^{-3}	0.193×10^{-2}	0.379×10^{-2}	0.648×10^{-2}

present about 75 seconds, would therefore reduce practically in proportional form the total computing time. This reduction is believed possible and is part of a future program of work.

As should be expected, the closer the spheroid is to the free surface, the larger the number of coefficients that must be kept in order to obtain the limiting values to a given accuracy. Most likely, not all products ${}^{m}_{n}B^{m'}_{n'}$, $A^{m'}_{n'}$ in (44), corresponding to a given number of equations, need be kept in order to obtain the solution to that same accuracy. Moreover, not all coefficients A^{m}_{n}, up to a certain one in the order in which they appear in expansion (34), need be computed in order to obtain, for example, the wave resistance, also to that same accuracy, since not all terms in the summation in (47) contribute significantly to the final value. It is difficult, however, to estimate beforehand the error associated with such approximations, a point of practical interest since a significant reduction in computing time could result thereof; no attempt has been made in this work to obtain such estimates. It is interesting to note that when only A^{0}_{1} is kept (one equation), despite the fact that the resulting approximation for A^{0}_{1} is close to the limit value, the correction for the wave resistance is rather small, since it turns out that some of the coefficients left out contribute rather significantly to its final value. Moreover, keeping A^{1}_{1} in addition to A^{0}_{1}, does not improve the approximation to the wave resistance, which remains unchanged to the three significant figures given in Table 5.

A set of equations essentially equivalent to (44) for determining the density of the source distribution on the surface of the spheroid

was obtained by Bessho (1957), using an entirely independent derivation. Bessho considered first an ellipsoid with three unequal axes, and was thus able to use in his solution several results on Lame's functions given by Hobson (1931, Chap. 11). The coefficients of his set of equations, however, appear to be incorrect, possibly because of typographical errors, and his numerical evaluations are rather inaccurate. His approximation to the solution of the infinite set of equations, which contains only six of the coefficients of the system, all belonging to the first five equations, introduces, in the light of the present numerical results, a significant error. Moreover, rather than using the singular solution for a source in the form of a double integral and a single integral as done in the present work, the coefficients of the infinite system of equations are expressed in terms of Rayleigh's "fictitious viscosity" μ, and are not therefore in a form suitable for numerical evaluation; the calculations are performed using asymptotic expansions, with which large errors are associated unless the Froude number is small enough. For a Froude number $U/\sqrt{2gc} = 0.395$, the relative depth of submergence used in the present calculations, $d/c = 0.5$, and a ratio $a/c = \sqrt{17/4} \sim 1.03$, Bessho obtains $R/4\pi\rho U^2c^2 = 7.165\text{x}10^{-4}$, a 146% increase over the value corresponding to Havelock's approximation, $2.91\text{x}10^{-4}$. For Froude number 0.4, Table 5 shows a 34% increase over the value corresponding to Havelock's approximation for $a/c = 1.02$, and a 90% increase for $a/c = 1.04$.

The relative density σ/U of the source distribution on the surface of the spheroid, corresponding to the successive approximations obtained keeping increasing numbers of equations in (44), for $a/c = 1.02$, is presented in Figs. 1, 2, and 3. These figures show that, although

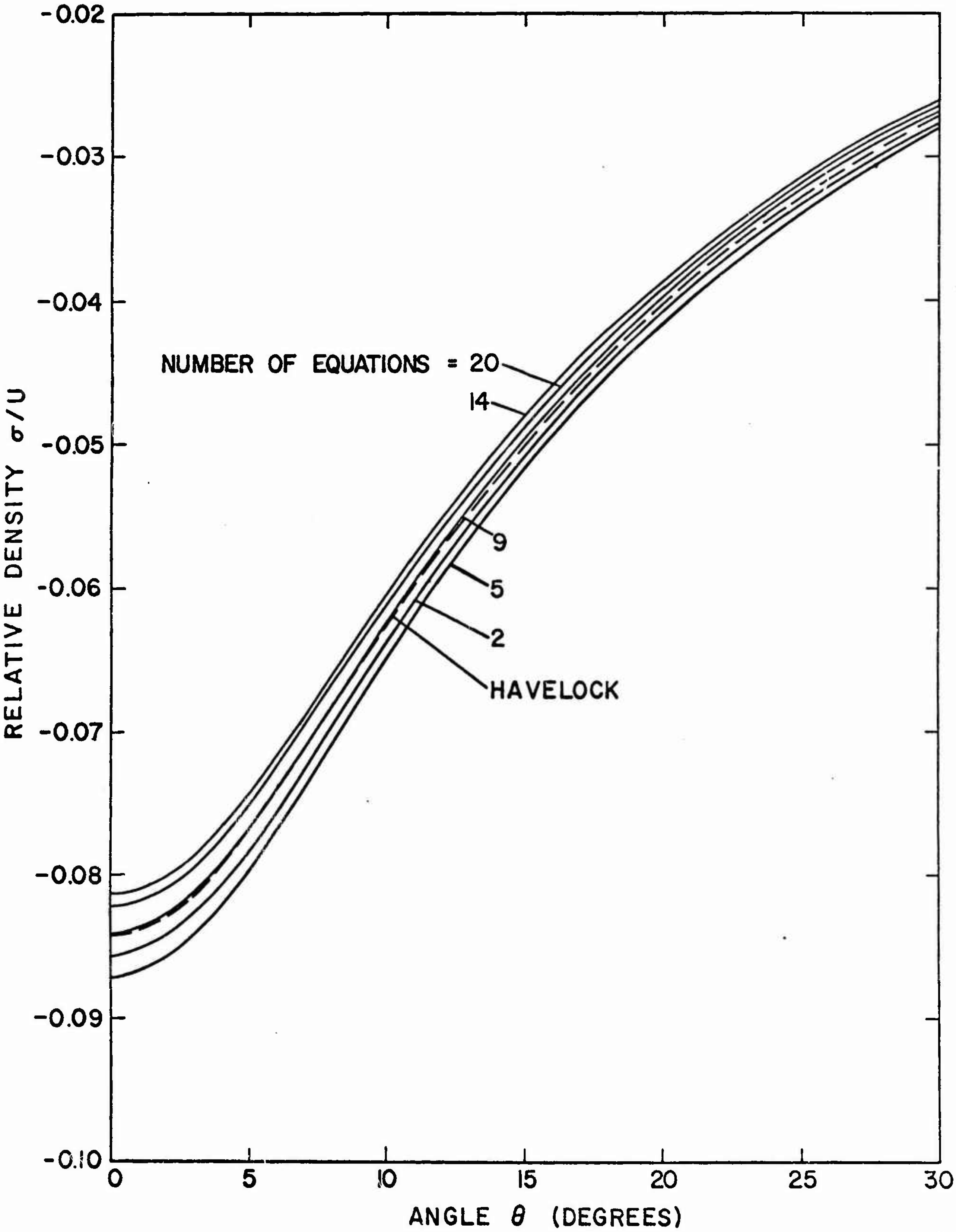

Figure 1. Density of source distribution along a horizontal meridian plane near rear of spheroid (a/c = 1.02)

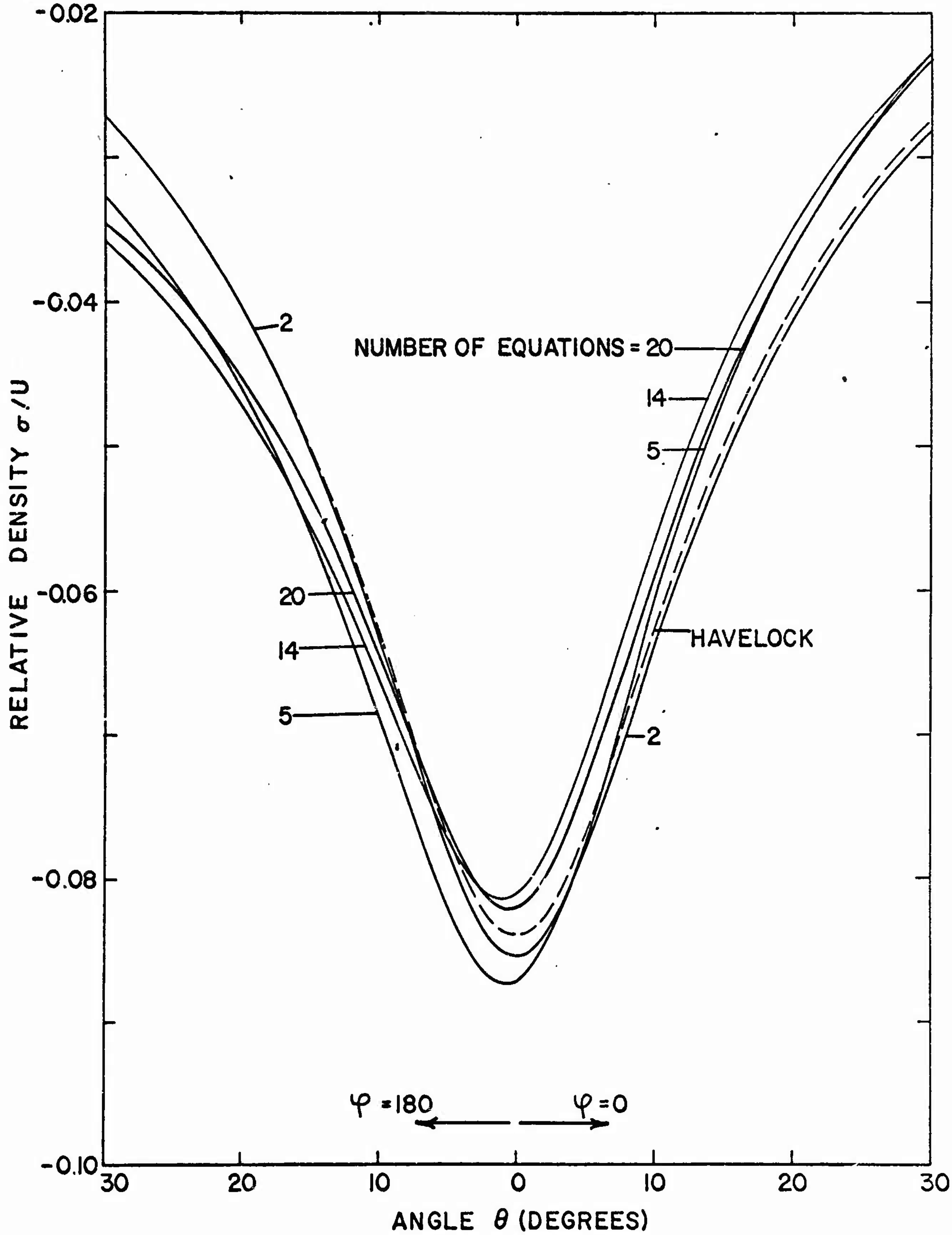

Figure 2. Density of source distribution along a vertical meridian plane near rear of spheroid (a/c = 1.02)

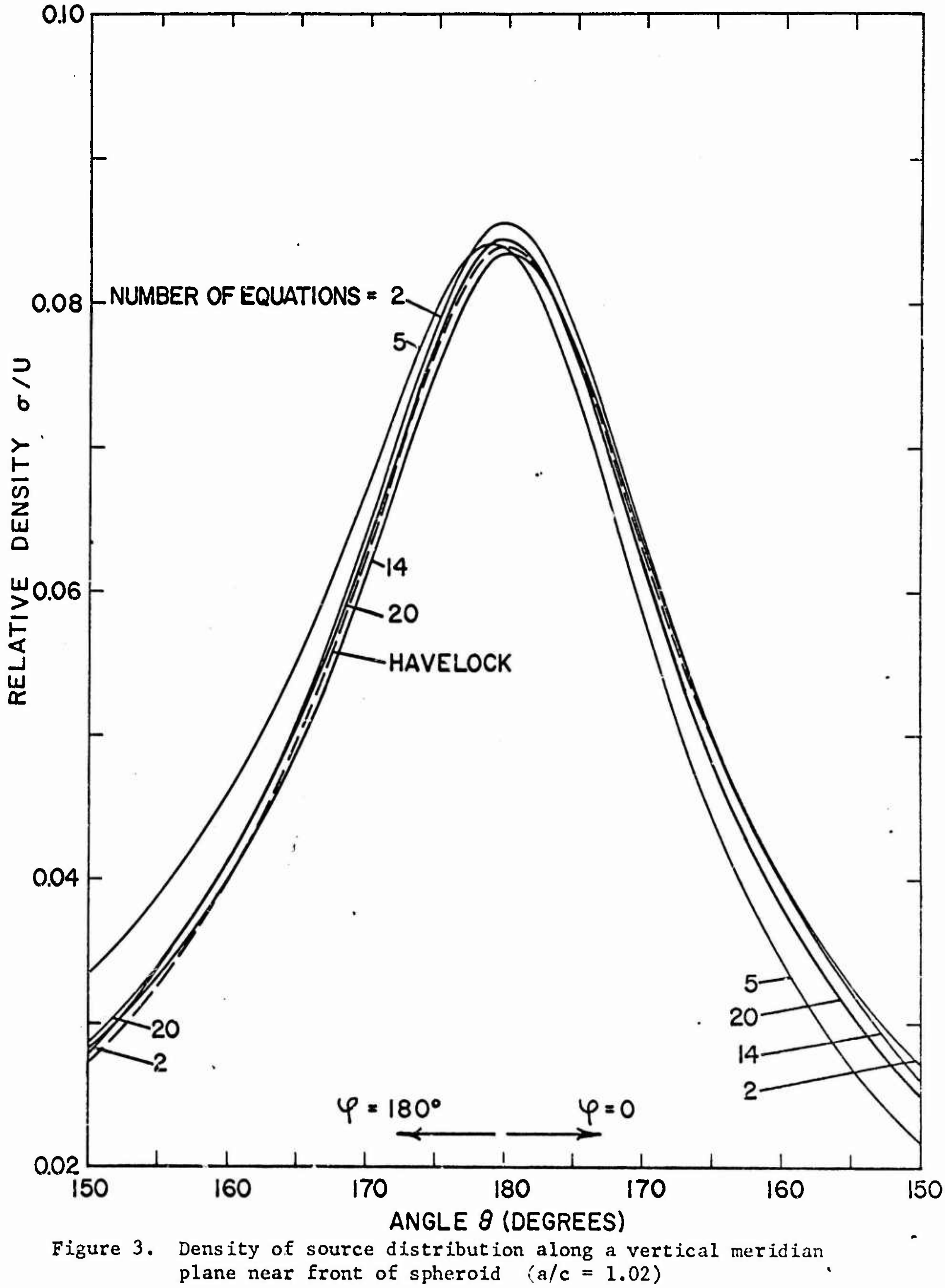

Figure 3. Density of source distribution along a vertical meridian plane near front of spheroid ($a/c = 1.02$)

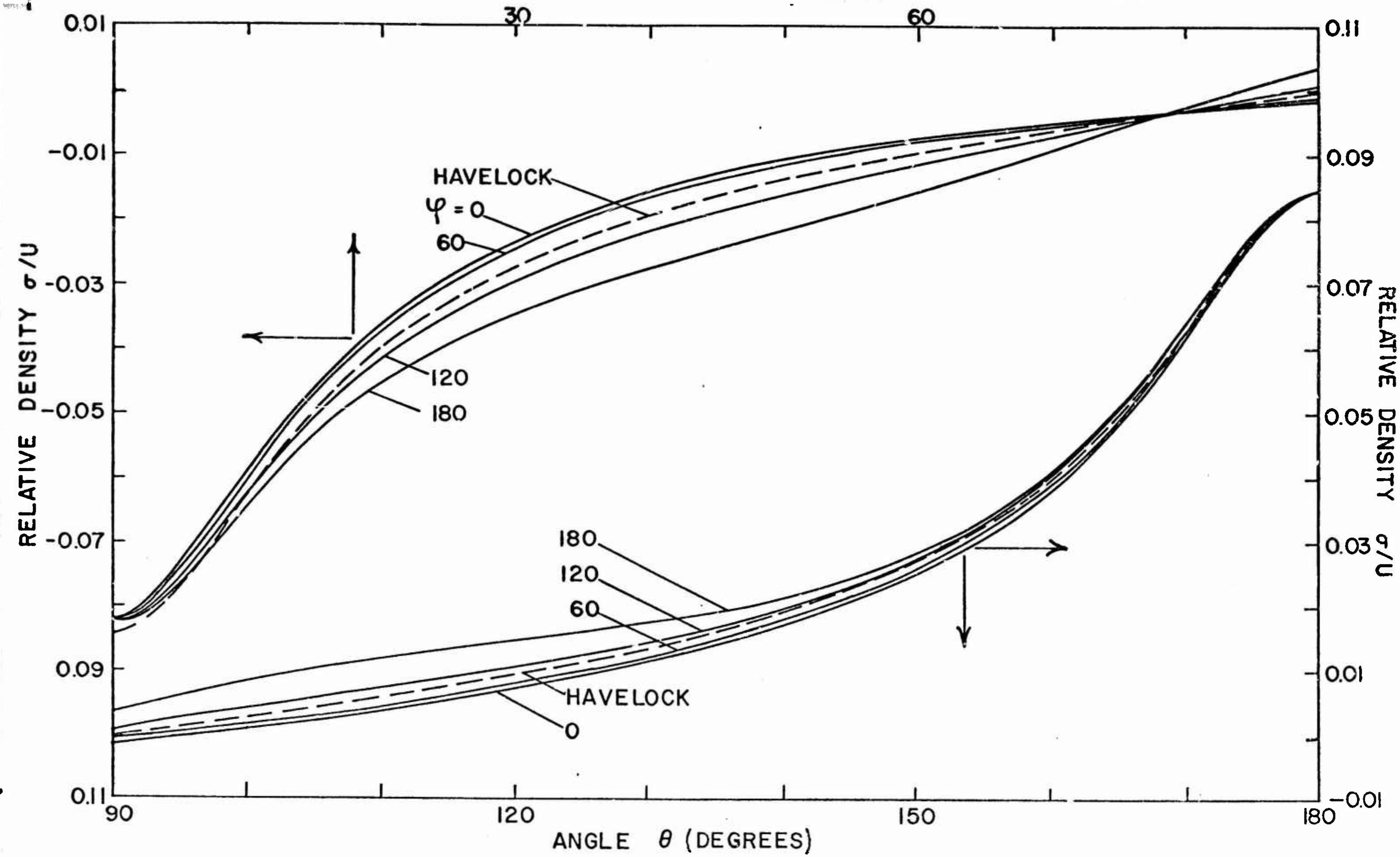

Figure 4. Density of source distribution along meridian planes of the spheroid (a/c = 1.02)

for an integrated quantity like the wave resistance, twenty equations are enough to obtain the solution, the convergence is slower for the surface density. Since the limiting values of most of the coefficients A_n^m in Table 2 have been practically attained (with the probable exception of those in the last row), this implies that some of the coefficients left out have a significant effect on the value of the density of the source distribution. It is interesting to note that the surface density oscillates about the final value; the approximation obtained keeping five equations in (44) is worse than Havelock's approximation over a large portion of the surface of the spheroid. The approximation obtained keeping twenty equations in (44) is depicted in Fig. 4, where the relative density is plotted along meridian planes of the spheroid, together with Havelock's approximation.

The solution obtained in this work is an "exact" solution within the theory of infinitesimal waves. The determination, for the same particular shape, of the error associated with the linearization of the free surface boundary condition, using the approximation techniques leading to (25) for the second-order contribution to the solution of the nonlinear problem, constitutes a natural continuation of the present study. For the circular cylinder, Tuck (1966) has shown that the error associated with the failure to satisfy the free-surface boundary condition is larger than that associated with the failure to satisfy the boundary condition on the surface of the body. The circular cylinder, however, is a shape of little practical interest, and the corresponding result for slender prolate spheroids should prove a valuable contribution to the general knowledge in this area.

CONCLUSIONS

The flow about a submerged prolate spheroid in axial horizontal motion beneath a free surface has been treated in this work and an "exact" solution within the theory of infinitesimal waves has been obtained. Comparison of this solution with Havelock's approximation reveals that significant errors are associated with the latter, in which the boundary condition on the surface of the body is not satisfied exactly. For a prolate spheroid with slenderness ratio (the ratio of major to minor axis) slightly larger than five, a focal distance twice the depth of submergence, and a Froude number (defined with respect to the distance between foci) of 0.4, the wave resistance is larger than Havelock's by about 34%. For slenderness ratios of 3.64 and 2.40, the same relative depth of submergence, and the same Froude number, the corrections are as much as 90% and 368%, respectively, of Havelock's approximation (the spheroid corresponding to the latter slenderness ratio is very close to piercing the free surface).

The successive approximations computed in order to obtain the final values of the coefficients of the required expansions in spheroidal harmonics, and the corresponding values of the surface density of the source distribution and the wave resistance, show that the corrections are small when only a few terms in the expansions are retained. The convergence is slower for the surface distribution, which

oscillates about the solution, and the approximation obtained by keeping only a few terms in the expansion is, for some portions of the spheroid, farther from the solution than Havelock's approximation.

An infinite set of equations, essentially equivalent to that obtained in this work for determining a source distribution on the surface of the spheroid which satisfies exactly the boundary condition on its surface, was obtained by Bessho using an entirely independent derivation. The coefficients of Bessho's system of equations, however, appear to be incorrect, possibly because of typographical errors, and his numerical evaluations are rather inaccurate. The value of the wave resistance obtained by Bessho, for a Froude number of 0.395, a focal distance equal to twice the depth of submergence, and a slenderness ratio of 4.17, exceeds Havelock's approximation by 146%; according to the numerical evaluations reported here, the correction should instead be in the neighborhood of 60 to 65% of Havelock's value.

The solution obtained in this work is an "exact" solution within the theory of infinitesimal waves. The determination of the error associated with the linearization of the free-surface boundary condition constitutes a natural continuation of the present study.

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APPENDIX A

EXPRESSION FOR SURFACE DISTRIBUTION IN SPHEROIDAL COORDINATES

The surface density σ_n^m corresponding to the external potential

$$S_{n\,ext}^{m} = P_n^m(\cos\theta)\cos m\varphi\,\frac{Q_n^m(\cosh\eta)}{Q_n^m(\cosh\eta_0)}$$

is derived in this Appendix. The corresponding internal potential is

$$S_{n\,int}^{m} = P_n^m(\cos\theta)\cos m\varphi\,\frac{P_n^m(\cosh\eta)}{P_n^m(\cosh\eta_0)}$$

We have

$$4\pi\sigma_n^m = \frac{\partial}{\partial n}S_{n\,ext}^{m} - \frac{\partial}{\partial n}S_{n\,int}^{m}\Big|_{\eta=\eta_0}$$

that is,

$$4\pi\sigma_n^m = P_n^m(\cos\theta)\cos m\varphi\left\{\frac{\dot{Q}_n^m(\cosh\eta_0)}{Q_n^m(\cosh\eta_0)} - \frac{\dot{P}_n^m(\cosh\eta_0)}{P_n^m(\cosh\eta_0)}\right\}\frac{\sinh\eta_0}{c(\cosh^2\eta_0-\cos^2\theta)^{\frac{1}{2}}}$$

since the length ds of the element of arc in spheroidal coordinates is given by

$$(ds)^2 = c^2(\cosh^2\eta-\cos^2\theta)\left\{(d\eta)^2+(d\theta)^2\right\} + c^2\sinh^2\eta\,\sin^2\theta\,(d\varphi)^2$$

We have moreover (see Appendix B)

$$(\mu^2-1)\left\{ Q_n^m(\mu)\,\dot{P}_n^m(\mu) - P_n^m(\mu)\,\dot{Q}_n^m(\mu) \right\} = (-1)^m \frac{(n+m)!}{(n-m)!}$$

The final result is then

$$\sigma_n^m = \frac{(-1)^{m+1}}{4\pi}\,\frac{(n+m)!}{(n-m)!}\,\frac{1}{c\,\sinh\eta_0(\cosh^2\eta_0-\cos^2\theta)^{\frac{1}{2}}}\,\frac{P_n^m(\cos\theta)\cos m\varphi}{P_n^m(\cosh\eta_0)\,Q_n^m(\cosh\eta_0)}$$

APPENDIX B

PROOF OF A RELATION INVOLVING LEGENDRE FUNCTIONS

The relation

$$(\mu^2-1)\left\{ Q_n^m(\mu)\,\dot{P}_n^m(\mu) - P_n^m(\mu)\,\dot{Q}_n^m(\mu)\right\} = (-1)^m \frac{(n+m)!}{(n-m)!}$$

required in Appendix A, is here derived. Since

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\,P_n^m(\mu)}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\cdot P_n^m(\mu) = 0$$

and

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\,Q_n^m(\mu)}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} Q_n^m(\mu) = 0$$

we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\left[Q_n^m(\mu)\,\frac{d\,P_n^m(\mu)}{d\mu} - P_n^m(\mu)\,\frac{d\,Q_n^m(\mu)}{d\mu}\right]\right\} = 0$$

and therefore

$$(1-\mu^2)\left\{ Q_n^m(\mu)\,\dot{P}_n^m(\mu) - P_n^m(\mu)\,\dot{Q}_n^m(\mu)\right\} = \text{constant.}$$

---

*This formula is obtained by Lamb in his Hydrodynamics (p. 142) from the relationship

$$Q_n^m(z) = (-1)^m \frac{(n+m)!}{(n-m)!}\, P_n^m(z) \int_z^\infty \frac{dz}{\left\{P_n^m(z)\right\}^2 (z^2-1)}$$

the proof of which, however, is not given.

To evaluate this constant, we use the expansions

$$P_n^m(\mu) = \frac{1}{2^n n!} (\mu^2-1)^{\frac{1}{2}m} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2-1)^n$$

$$= \frac{(2n)!}{2^n n!(n-m)!} (\mu^2-1)^{\frac{1}{2}m} \left\{ \mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} \mu^{n-m-2} + \ldots \right\}$$

(Hobson, p. 91), and

$$Q_n^m(\mu) = (-1)^m \frac{2^n n!(n+m)!}{(2n+1)!} (\mu^2-1)^{\frac{1}{2}m}$$

$$\cdot \left\{ \frac{1}{\mu^{n+m+1}} + \frac{(n+m+1)(n+m+2)}{2(2n+3)} \frac{1}{\mu^{n+m+3}} + \ldots \right\}$$

(Hobson, p. 108), which hold when μ is not real and between 1 and -1. The result follows immediately.

APPENDIX C

EXPRESSION OF $e^{\alpha x+\beta y+\gamma z}$, $\alpha^2+\beta^2+\gamma^2 = 0$ IN SPHEROIDAL HARMONICS

We require the expansion of the function

$$e^{\alpha x+\beta y+\gamma z}, \quad \alpha^2+\beta^2+\gamma^2 = 0 \tag{1}$$

in a series of spheroidal harmonics in the region interior to an arbitrary spheroid centered at the origin. Here the x axis is assumed to be the axis of the spheroidal coordinate system.

In the particular case

$$\alpha = k, \quad \beta = ik \cos t, \quad \gamma = ik \sin t \tag{2}$$

this expansion can be obtained without much difficulty using the relationship

$$P_n\left(\frac{x+iy\cos t+iz\sin t}{c}\right) = P_n(\cosh\eta)\,P_n(\cos\theta)$$
$$+ 2\sum_{m=1}^{n} i^{-m}\frac{(n-m)!}{(n+m)!}P_n^m(\cosh\eta)\,P_n^m(\cos\theta)\cos m(\varphi-t) \tag{3}$$

(Hobson 1931, p. 414) and expanding

$$e^{kx+ik(y\cos t+z\sin t)} = e^{kc\frac{x+i(y\cos t+z\sin t)}{c}} = e^{kcu}$$

in a series of Legendre polynomials in the variable u,

$$e^{kcu} = \sum_{n=0}^{\infty} a_n P_n(u) \tag{4}$$

The coefficients a_n are given by

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} e^{kcu} P_m(u)\,du = \frac{2m+1}{2} J_{m+\frac{1}{2}}(-ikc)\sqrt{\frac{2\pi}{(-ikc)}}\,(-i)^{-m}$$

or

$$a_m = (2m+1)\sqrt{\frac{\pi}{2\Delta}}\, I_{m+\frac{1}{2}}(\Delta), \quad \Delta = kc \tag{5}$$

Making use of (3), (4), and (5), the expansion is obtained in the form

$$\sum_{n=0}^{\infty} C_n P_n(\cos\theta) P_n(\cosh\eta) + \sum_{n=0}^{\infty}\sum_{m=1}^{n} (C_{n,m}\cos m\varphi + B_{n,m}\sin m\varphi) P_n^m(\cos\theta) P_n^m(\cosh\eta) \tag{6}$$

where

$$C_n = (2n+1)\sqrt{\frac{\pi}{2\Delta}}\, I_{n+\frac{1}{2}}(\Delta)$$

$$C_{n,m} = i^{-m}\, 2(2n+1)\frac{(n-m)!}{(n+m)!}\sqrt{\frac{\pi}{2\Delta}}\cos mt\; I_{n+\frac{1}{2}}(\Delta) \tag{7}$$

$$B_{n,m} = i^{-m}\, 2(2n+1)\frac{(n-m)!}{(n+m)!}\sqrt{\frac{\pi}{2\Delta}}\sin mt\; I_{n+\frac{1}{2}}(\Delta)$$

Unfortunately this derivation cannot be applied to the case of interest in the present problem, the expansion of the function

$$e^{ky+ik(x\cos t + z\sin t)} \tag{8}$$

since Hobson's result applies to

$$P_m\left(\frac{y + ix\cos t + iz\sin t}{c}\right)$$

only if the y axis is the axis of the spheroidal coordinate system. The technique to be used here to obtain the expansion of (8) is based again on results of Hobson and yields a more general expansion of (1).

Let $f(x,y,z)$ be a potential function over a bounded region R containing the origin. Choose any spheroid $\eta = \eta_1$ contained in R, η_1 being arbitrary. We obtain the expansion of $f(x,y,z)$ in spheroidal harmonics by first expanding the value of this function on the surface S of the spheroid in a series of surface spherical harmonics,

$$F(\theta,\varphi) = f(c\cosh\eta_1\cos\theta,\ c\sinh\eta_1\sin\theta\cos\varphi,\ c\sinh\eta_1\sin\theta\sin\varphi)$$

$$= \sum_{n=0}^{\infty} C'_n P_n(\cos\theta) + \sum_{n=0}^{\infty}\sum_{m=1}^{n}\left(C'_{n,m}\cos m\varphi + B'_{n,m}\sin m\varphi\right) P_n^m(\cos\theta) \tag{9}$$

and then writing

$$f(x,y,z) = \sum_{n=0}^{\infty} C'_n P_n(\cos\theta)\,\frac{P_n(\cosh\eta)}{P_n(\cosh\eta_1)}$$

$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(C'_{n,m}\cos m\varphi + B'_{n,m}\sin m\varphi\right) P_n^m(\cos\theta)\,\frac{P_n^m(\cosh\eta)}{P_n^m(\cosh\eta_1)} \tag{10}$$

(Hobson, pp. 417 et seqq.). We assume the series expansion (9) to be uniformly and absolutely convergent over the surface S. Its coefficients are given by

$$C'_n = \frac{2n+1}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(\theta,\varphi)\, P_n(\cos\theta)\, \sin\theta \, d\theta \, d\varphi$$

$$C'_{n,m} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} \int_0^{2\pi} F(\theta,\varphi)\, P_n^m(\cos\theta)\, \cos m\varphi \, \sin\theta \, d\theta \, d\varphi \qquad (11)$$

$$B'_{n,m} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} \int_0^{2\pi} F(\theta,\varphi)\, P_n^m(\cos\theta)\, \sin m\varphi \, \sin\theta \, d\theta \, d\varphi$$

It should be noticed that the expansion for the space internal to an ellipsoid has been used. The expansion for the external space, which involves $Q_n^m(\cosh\eta)$ instead of $P_n^m(\cosh\eta)$, cannot be used since the function to be expanded is not regular at infinity and therefore Hobson's theorems are not valid.

Put

$$\begin{aligned} x' &= \cos\theta, \\ y' &= \sin\theta \cos\varphi, \\ z' &= \sin\theta \sin\varphi. \end{aligned} \qquad (12)$$

The value of the function $f(x,y,z)$ on the surface of the spheroid is then given by

$$f(ax',by',bz')$$

where use has been made of the relationships

$$a = c\cosh\eta_1, \quad b = c\sinh\eta_1$$

and the repeated integrals in (11) become

$$\int_\sigma f(ax',by',bz')\, Y_n^m(\;\;,y',z')\, d\sigma \qquad (13)$$

the integration being taken over the surface of the unit sphere $x'^2+y'^2+z'^2 = 1$, and Y_n^m denoting any of the $2n+1$ independent ordinary spherical harmonics of degree n:

$$r'^n P_n^m(\cos\theta), \quad r'^n P_n^m(\cos\theta)\cos m\varphi, \quad r'^n P_n^m(\cos\theta)\sin m\varphi,$$

$$m = 1, 2, \ldots, n,$$

with
$$r'^2 = x'^2+y'^2+z'^2$$

These functions, as is well known, are homogeneous polynomials in x', y', and z'.

To evaluate the double integral (13) we use a theorem of Hobson. We have (Hobson, p. 161)

$$\int_\sigma f(ax', by', bz')\, Y_m(x', y', z')\, d\sigma = 4\pi \frac{2^m m!}{(2m+1)!}$$

$$\left\{1 + \frac{\nabla^2}{2(2m+3)} + \frac{\nabla^4}{2.4(2m+3)(2m+5)} + \cdots\right\} Y_m\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right) f(ax', by', bz') \quad (14)$$

where
$$\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$

and, on the right side, x', y', and z' are put equal to zero after the operations are performed. The operator $Y_n^m\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$ is obtained replacing x', y', and z' in the homogeneous polynomial $Y_n^m(x',y',z')$ by the differential operators $\frac{\partial}{\partial x'}$, $\frac{\partial}{\partial y'}$, and $\frac{\partial}{\partial z'}$,

respectively.

It can be easily shown that the conditions of the theorem are satisfied for our function,

$$e^{\alpha a x' + \beta b y' + \gamma b z'}$$

We have

$$\left\{ 1 + \frac{\nabla^2}{2(2m+3)} + \frac{\nabla^4}{2.4\,(2m+3)(2m+5)} + \cdots \right\} e^{\alpha a x' + \beta b y' + \gamma b z'}$$

$$= \left\{ 1 + \frac{\Delta^2}{2(2m+3)} + \frac{\Delta^4}{2.4\,(2m+3)(2m+5)} + \cdots \right\} e^{\alpha a x' + \beta b y' + \gamma b z'}$$

where

$$\Delta^2 = \alpha^2 a^2 + (\beta^2 + \gamma^2) b^2 = \alpha^2 (a^2 - b^2) = \alpha^2 c^2$$

Since, moreover (see Appendix D),

$$Y_n^m\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right) e^{\alpha a x' + \beta b y' + \gamma b z'} = \frac{(-1)^m}{2} \left\{ \begin{matrix} 1 \\ i \end{matrix} \right\} \alpha^{n-m} c^n P_n^m(\cosh \eta_1)$$

$$\left\{ (\beta - i\gamma)^m \pm (\beta + i\gamma)^m \right\} e^{\alpha a x' + \beta b y' + \gamma b z'}$$

where the factor i and the minus sign within the braces correspond to the n harmonics containing $\sin m\varphi$, and the factor 1 and the plus sign to the remaining $(n+1)$ harmonics, we obtain finally

$$\int_0^{\pi}\int_0^{2\pi} e^{\alpha a\cos\theta+\beta b\sin\theta\cos\varphi+\gamma b\sin\theta\sin\varphi}\, P_n^m(\cos\theta)\cos m\varphi\,\sin\theta\, d\theta\, d\varphi$$

$$= 2\pi(-1)^m \frac{2^n n!}{(2n+1)!}\,\alpha^{n-m} c^n P_n^m(\cosh\eta_1)\left\{(\beta-i\gamma)^m+(\beta+i\gamma)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\cdots\right\} \tag{15}$$

and

$$\int_0^{\pi}\int_0^{2\pi} e^{\alpha a\cos\theta+\beta b\sin\theta\cos\varphi+\gamma b\sin\theta\sin\varphi}\, P_n^m(\cos\theta)\sin m\varphi\,\sin\theta\, d\theta\, d\varphi$$

$$= 2\pi(-1)^m \frac{2^n n!}{(2n+1)!}\, i\,\alpha^{n-m} c^n P_n^m(\cosh\eta_1)\left\{(\beta-i\gamma)^m-(\beta+i\gamma)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\cdots\right\} \tag{16}$$

If we write the required expansion in the form (6) the coefficients are given by

$$C_n = \frac{2^n n!}{(2n)!}\,\alpha^n c^n\left\{1+\frac{\Delta^2}{2(2n+3)}+\frac{\Delta^4}{2.4(2n+3)(2n+5)}+\cdots\right\}$$

$$C_{n,m} = (-1)^m \frac{2^n n!}{(2n)!}\frac{(n-m)!}{(n+m)!}\,\alpha^{n-m} c^n\left\{(\beta-i\gamma)^m+(\beta+i\gamma)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\cdots\right\} \tag{17}$$

$$B_{n,m} = (-1)^m \frac{2^n n!}{(2n)!}\frac{(n-m)!}{(n+m)!}\, i\,\alpha^{n-m} c^n\left\{(\beta-i\gamma)^m-(\beta+i\gamma)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\cdots\right\}$$

where

$$\Delta^2 = \alpha^2 c^2$$

In the particular case of interest to this work, we have

$$\alpha = ik\cos t, \quad \beta = k, \quad \gamma = ik\sin t \tag{18}$$

Equations (15) and (16) become then

$$\int_0^{\pi}\int_0^{2\pi} e^{kb\sin\theta\cos\varphi + ik(a\cos\theta\cos t + b\sin\theta\sin\varphi\sin t)} P_n^m(\cos\theta)\cos m\varphi\sin\theta\, d\theta\, d\varphi$$

$$= 2\pi(-1)^m \frac{2^m m!}{(2m+1)!} i^{n-m} P_n^m(\cosh\eta_1)(kc\cos t)^m \left\{(\sec t+\tan t)^m + \frac{1}{(\sec t+\tan t)^m}\right\}\left\{1 - \frac{\Delta^2}{2(2m+3)} + \cdots\right\} \quad (19)$$

and

$$\int_0^{\pi}\int_0^{2\pi} e^{kb\sin\theta\cos\varphi + ik(a\cos\theta\cos t + b\sin\theta\sin\varphi\sin t)} P_n^m(\cos\theta)\sin m\varphi\sin\theta\, d\theta\, d\varphi$$

$$= 2\pi(-1)^m \frac{2^m m!}{(2m+1)!} i^{n-m+1} P_n^m(\cosh\eta_1)(kc\cos t)^m \left\{(\sec t+\tan t)^m - \frac{1}{(\sec t+\tan t)^m}\right\}\left\{1 - \frac{\Delta^2}{2(2m+3)} + \cdots\right\} \quad (20)$$

where we have now put

$$\Delta = kc\cos t$$

Since

$$J_{m+\frac{1}{2}}(z) = \frac{\left(\frac{z}{2}\right)^{m+\frac{1}{2}}}{\Gamma\left(m+\frac{3}{2}\right)}\left(1 - \frac{z^2}{2(2m+3)} + \frac{z^4}{2.4(2m+3)(2m+5)} - \cdots\right)$$

we have

$$\frac{2^m m!}{(2m)!}(kc\cos t)^m\left(1 - \frac{\Delta^2}{2(2m+3)} + \cdots\right) = \frac{2^m m!}{(2m)!}\Gamma\left(m+\frac{3}{2}\right) 2^{m+\frac{1}{2}} \frac{1}{\sqrt{\Delta}} J_{m+\frac{1}{2}}(\Delta)$$

Moreover,

$$\frac{2^m m!}{(2m)!} 2^{m+1}\Gamma\left(m+\frac{3}{2}\right) = (2m+1)\sqrt{\pi}$$

and therefore

$$\frac{2^n n!}{(2n)!}\,\Delta^n\left(1-\frac{\Delta^2}{2(2n+2)}+\cdots\right)=\sqrt{\frac{\pi}{2\Delta}}\,(2n+1)\,J_{n+\frac{1}{2}}(\Delta) \tag{21}$$

Making use of (21), we can write (19) and (20) in the more compact form

$$\int_0^{\pi}\int_0^{2\pi} e^{kb\sin\theta\cos\varphi+ik(a\cos\theta\cos t+b\sin\theta\sin\varphi\sin t)}\,P_n^m(\cos\theta)\cos m\varphi\,\sin\theta\,d\theta\,d\varphi$$

$$=2\pi\, i^{n+m}\,P_n^m(\cosh\eta_1)\left\{(\sec t+\tan t)^m+\frac{1}{(\sec t+\tan t)^m}\right\}\sqrt{\frac{\pi}{2\Delta}}\,J_{n+\frac{1}{2}}(\Delta) \tag{22}$$

and

$$\int_0^{\pi}\int_0^{2\pi} e^{kb\sin\theta\cos\varphi+ik(a\cos\theta\cos t+b\sin\theta\sin\varphi\sin t)}\,P_n^m(\cos\theta)\sin m\varphi\,\sin\theta\,d\theta\,d\varphi$$

$$=2\pi\, i^{n+m+1}\,P_n^m(\cosh\eta_1)\left\{(\sec t+\tan t)^m-\frac{1}{(\sec t+\tan t)^m}\right\}\sqrt{\frac{\pi}{2\Delta}}\,J_{n+\frac{1}{2}}(\Delta) \tag{23}$$

The coefficients of expansion (6) for the function

$$e^{ky+ik(x\cos t+z\sin t)}$$

are then given by

$$C_n = i^n(2n+1)\sqrt{\frac{\pi}{2\Delta}}\;J_{n+\frac{1}{2}}(\Delta)$$

$$C_{n,m}=i^{n+m}\frac{(n-m)!}{(n+m)!}(2n+1)\sqrt{\frac{\pi}{2\Delta}}\left\{(\sec t+\tan t)^m+\frac{1}{(\sec t+\tan t)^m}\right\}J_{n+\frac{1}{2}}(\Delta) \tag{24}$$

$$B_{n,m}=i^{n+m+1}\frac{(n-m)!}{(n+m)!}(2n+1)\sqrt{\frac{\pi}{2\Delta}}\left\{(\sec t+\tan t)^m-\frac{1}{(\sec t+\tan t)^m}\right\}J_{n+\frac{1}{2}}(\Delta)$$

As a check on the validity of the expansions here derived, we reobtain the coefficients corresponding to the particular case (2) using the general expressions (17). We have

$$C_n = \frac{2^n n!}{(2n)!}\,\Delta^n\left\{1+\frac{\Delta^2}{2(2n+3)}+\frac{\Delta^4}{2.4(2n+3)(2n+5)}+\dots\right\}$$

$$C_{n,m} = (-1)^m\frac{2^n n!}{(2n)!}\,\frac{(n-m)!}{(n+m)!}\,\Delta^n\left\{(i\cos t+\sin t)^m+(i\cos t-\sin t)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\dots\right\} \quad (25)$$

$$B_{n,m} = (-1)^m\frac{2^n n!}{(2n)!}\,\frac{(n-m)!}{(n+m)!}\,i\,\Delta^n\left\{(i\cos t+\sin t)^m-(i\cos t-\sin t)^m\right\}\left\{1+\frac{\Delta^2}{2(2n+3)}+\dots\right\}$$

where now

$$\Delta = kc$$

Since

$$(i\cos t \pm \sin t)^m = i^m(\cos t \mp i\sin t)^m = i^m(\cos mt \mp i\sin mt)$$

and

$$\frac{2^n n!}{(2n)!}\,\Delta^n\left\{1+\frac{\Delta^2}{2(2n+3)}+\dots\right\} = \sqrt{\frac{\pi}{2\Delta}}\,(2n+1)\,I_{n+\frac{1}{2}}(\Delta)$$

where $I_{n+\frac{1}{2}}$ denotes the modified Bessel function of the first kind of order $(n+\frac{1}{2})$, the coefficients (25) can also be expressed in the form (7) as we wanted to show.

APPENDIX D

EXPRESSION FOR $Y_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e^{\alpha ax+\beta by+\gamma bz}$

We shall prove that

$$Y_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e^{\alpha ax+\beta by+\gamma bz} = \frac{(-1)^m}{2}\left\{\begin{matrix}1\\ i\end{matrix}\right\} \alpha^{n-m} c^n P_n^m(\cosh\eta_1)$$

$$\left\{(\beta-i\gamma)^m \pm (\beta+i\gamma)^m\right\} e^{\alpha ax+\beta by+\gamma bz}$$

where $\alpha^2+\beta^2+\gamma^2 = 0$, $a = c\cosh\eta_1$, $b = c\sinh\eta_1$

and Y_n^m denotes any of the $2n+1$ independent spherical harmonics of degree n

$$r^n P_n(\cos\theta),\quad r^n P_n^m(\cos\theta)\cos m\varphi,\quad r^n P_n^m(\cos\theta)\sin m\varphi,$$

$$m = 1, 2, \ldots, n$$

with $$r^2 = x^2 + y^2 + z^2$$

The factor i and the minus sign within the braces correspond to the n harmonics containing $\sin m\varphi$; the factor 1 and the plus sign to the remaining $(n+1)$ harmonics.

For the proof of this formula we make use again of a result put forward by Hobson. We have (Hobson, p. 93)

$$P_n^m(\cos\theta) = (-1)^m \frac{(n+m)!}{2^m m!(n-m)!} \sin^m\theta \left\{ \cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{2(2m+2)} \cos^{n-m-2}\theta \sin^2\theta \right.$$

$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4(2m+2)(2m+4)} \cos^{n-m-4}\theta \sin^4\theta - \dots \right\} \qquad (1)$$

Let us write, for shortness,

$$\xi = y+iz, \qquad \eta = y-iz. \qquad (2)$$

We have

$$\xi^m = (y^2+z^2)^{\frac{m}{2}} \cos m\varphi + i\,(y^2+z^2)^{\frac{m}{2}} \sin m\varphi$$

and

$$\eta^m = (y^2+z^2)^{\frac{m}{2}} \cos m\varphi - i\,(y^2+z^2)^{\frac{m}{2}} \sin m\varphi. \qquad (3)$$

On using (1) and (3) we obtain

$$r^n P_n^m(\cos\theta) \cos m\varphi = (-1)^m \frac{(n+m)!}{2^m m!(n-m)!} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2m+2)} x^{n-m-2} \xi\eta \right.$$

$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4(2m+2)(2m+4)} x^{n-m-4} (\xi\eta)^2 - \dots \right\} \frac{1}{2}(\xi^m + \eta^m), \quad m = 0, 1, \dots, n \qquad (4)$$

(Hobson, p. 137), and

$$r^n P_n^m(\cos\theta) \sin m\varphi = (-1)^m \frac{(n+m)!}{2^m m!(n-m)!} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2m+2)} x^{n-m-2} \xi\eta \right.$$

$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4(2m+2)(2m+4)} x^{n-m-4} (\xi\eta)^2 - \dots \right\} \frac{1}{2i}(\xi^m - \eta^m), \quad m = 1, 2, \dots, n \qquad (5)$$

We make use of these expressions to obtain

$$Y_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e^{\alpha ax+\beta by+\gamma bz}$$

Again, for shortness, put

$$f = \alpha ax+\beta by+\gamma bz$$

We have

$$\frac{\partial}{\partial x} e^{f} = \alpha a e^{f}. \tag{6}$$

$$\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right) e^{f} = b^2(\beta^2+\gamma^2)\, e^{f} = -b^2\alpha^2 e^{f} \tag{7}$$

and

$$\left\{\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^{m} \pm \left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)^{m}\right\} e^{f} = b^{m}\left\{(\beta+i\gamma)^{m} \pm (\beta-i\gamma)^{m}\right\} e^{f} \tag{8}$$

Moreover, since for all values of μ which are not on the real line segment $(-1,+1)$,

$$P_n^m(\mu) = \frac{(n+m)!}{2^m\, m!\,(n-m)!}(\mu^2-1)^{\frac{m}{2}}\left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2m+2)}\mu^{n-m-2}(1-\mu^2)\right.$$

$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4\,(2m+2)(2m+4)}\mu^{n-m-4}(1-\mu^2)^2 - \cdots\right\} \tag{9}$$

(Hobson, p. 93), we have

$$\left\{a^{n-m} + \frac{(n-m)(n-m-1)}{2(2m+2)}a^{n-m-2}b^2 + \cdots\right\}$$

$$= c^{n-m}\left\{(\cosh\eta_1)^{n-m} + \frac{(n-m)(n-m-1)}{2(2m+2)}(\cosh\eta_1)^{n-m-2}(\cosh^2\eta_1-1) + \cdots\right\}$$

$$= c^{n-m}\, P_n^m(\cosh\eta_1)\, \frac{2^m\, m!\,(n-m)!}{(n+m)!}\left(\frac{c}{b}\right)^m \tag{10}$$

When use is made of (6), (7), (8), and (10) the required result follows immediately.

APPENDIX E

COMPUTER PROGRAMS

```
C
C.............................................................
C
C     EVALUATION OF SINGLE INTEGRAL BY HERMITE-GAUSS QUADRATURE
C
C.............................................................
C
      DOUBLE PRECISION BESSJ, ARG, FR
      DIMENSION X(10),W(10),F(10),SINT(2000),IND(2000),
     1INDNUM(2000)
C
      READ (5,3) NQUAD,(X(I),W(I),I=1,NQUAD)
    3 FORMAT (I10/(2F30.15))
      WRITE (6,4) (X(I),W(I),I=1,NQUAD)
    4 FORMAT (1H1,16X,4HX(I),36X,4HW(I)//(1H ,E30.15,10X,E30.15))
C
      PI = 3.141593
C
      READ (5,7) ERR, LIMIT, NUMINT
    7 FORMAT (F20.10,2I10)
      READ (5,1) FR, DOC
    1 FORMAT (2F10.5)
C
      GCOU2 = 1.0/(2.0*FR**2)
      GDOU2 = GCOU2*DOC
      WRITE (6,2) FR, GCOU2, GDOU2, DOC, ERR
    2 FORMAT (1H1,21HFR = U/SQRT(2*G*C) = ,F10.5,10X,
     111HG*C/U**2 = ,F10.5,10X,11HG*D/U**2 = ,F10.5,10X,
     26HD/C = ,F10.5,10X/1H0,6HERR = ,F20.15)
C
      H = 1.0/ SQRT(2.0*GDOU2)
      FACT = 8.0*PI* EXP(-2.0*GDOU2)*H
C
      DO 29 J=1,NUMINT
C
      READ (5,19) N, M, NP, MP
   19 FORMAT (4I10)
      INDNUM(J) = 1000*N+100*M+10*NP+MP
      IND(J) = N+M+NP+MP
C
      IF (IND(J)/2*2-IND(J)) 15,16,15
   16 SINT(J) = 0.0
      GO TO 29
C
   15 WRITE (6,61) N, M, NP, MP
   61 FORMAT (///1H0,4HN = ,I2,10X,4HM = ,I2,10X,5HNP = ,I2,
     110X,5HMP = ,I2)
      SUM = 0.0
C
```

```
      DO 30 I=1,NQUAD
C
      XH = X(I)*H
      RC =  SQRT(1.0+XH**2)
      RM = (RC+XH)**M
      RMP = (RC+XH)**MP
      ARG = GCOU2*RC
      ER = ERR
      F(I) = 0.25*(RM+1.0/RM)*(RMP+1.0/RMP)*BESSJ(M,ARG,ER,LIMIT)
     1*BESSJ(MP,ARG,ER,LIMIT)/RC
   30 SUM = SUM+F(I)*W(I)
C
      SINT(J) = SUM*FACT
      WRITE (6,5) SINT(J), SUM, (I,F(I),I=1,NQUAD)
    5 FORMAT (1H0,7HSINT = ,F20.15,20X,6HSUM = ,F20.15/1H0,9X,
     11HI,20X,1HF//(1H ,I10,F30.15))
C
   29 CONTINUE
C
      WRITE (6,499) (SINT(I), INDNUM(I), I=1,NUMINT)
  499 FORMAT (1H1,17X,4HSINT,25X,4HNMNM///(1H0,10X,F20.15,10X,
     1I10))
      WRITE (7,599) (SINT(I), INDNUM(I), I=1,NUMINT)
  599 FORMAT (E20.10, 20X, I10)
      CALL EXIT
      END
C
C.....................................................................
C
C     FUNCTION BESSJ
C
C     BESSEL FUNCTION J OF ORDER  N+1/2
C
C.....................................................................
C
      DOUBLE PRECISION FUNCTION BESSJ(N,X,ERR,LIMIT)
      DOUBLE PRECISION X, ERR, PI, FACTOR, TERM, SUM, X2, DSIN,
     1DCOS, DSQRT, DABS, BESJ, FACT
      DIMENSION BESJ(100)
      PI = 3.1415926535898
      IF (X-5.0) 2,2,3
C
    3 NP1 = N+1
C
C     THE RECURRENCE RELATION
C               J(N+1,X)+J(N-1,X) = (2*N/X)*J(N,X)
C     IS USED
C
      IF (N-1) 13,11,16
```

```
   13 BESJ(1) = DSIN(X)
      GO TO 18
   11 BESJ(2) = DSIN(X)/X-DCOS(X)
      GO TO 18
   16 BESJ(1) = DSIN(X)
      BESJ(2) = DSIN(X)/X-DCOS(X)
      DO 19 I=3,NP1
      AI = I
   19 BESJ(I) = (2.0*AI-3.0)/X*BESJ(I-1)-BESJ(I-2)
   18 BESSJ = BESJ(NP1)*DSQRT(2.0/(PI*X))
      GO TO 101
C
    2 FACT = 1.0
C
C     THE EXPRESSION OF A BESSEL FUNCTION J AS A SERIES OF
C     ASCENDING POWERS OF THE ARGUMENT IS USED
C
      IF (N) 8,7,8
    8 DO 5 I = 1,N
      A2IP1 = 2*I+1
    5 FACT = FACT*X/A2IP1
    7 FACTOR = DSQRT(2.0/PI)*(X**(0.5))*FACT
      TERM = 1.0
      SUM = 1.0
      AN2 = 2*N
      X2 = X**2
      DO 10 I = 1,LIMIT
      AI2 = 2*I
      TERM = (-1.0)*TERM*X2/(AI2*(AN2+AI2+1.0))
      SUM = SUM+TERM
      IF (DABS(TERM)-DABS(ERR*SUM)) 12,12,10
   10 CONTINUE
      WRITE (6,113) TERM, SUM, X
  113 FORMAT (1H0,40HREQUIRED ACCURACY NOT MET IN LIMIT STEPS/1H ,
     17HTERM = ,F20.15,10X,6HSUM = ,F20.15,10X,4HX = ,F20.15)
   12 BESSJ = SUM*FACTOR
  101 RETURN
      END
```

```
C
C.......................................................................
C
C     EVALUATION OF SINGLE INTEGRAL BY SIMPSON S RULE
C
C.......................................................................
C
      EXTERNAL F
      COMMON N, M, NP, MP, GDOU2, GCOU2, ERRBES, LIMBES
      DIMENSION SINT(1000), IND(1000), INDNUM(1000)
C
      PI = 3.141593
C
      READ (5,11) ERR, ERRBES, LIMIT1, LIMIT2, LIMBES, NUMINT
   11 FORMAT (2F20.10,3I10/I10)
      READ (5,1) FR, DOC
    1 FORMAT (2F10.5)
C
      GCOU2 = 1.0/(2.0*FR**2)
      GDOU2 = GCOU2*DOC
      WRITE (6,2) FR, GCOU2, GDOU2, DOC, ERR
    2 FORMAT (1H1,21HFR = U/SQRT(2*G*C) = ,F10.5,10X,
     111HG*C/U**2 = ,F10.5,10X,11HG*D/U**2 = ,F10.5,10X,
     26HD/C = ,F10.5,10X/1HC,6HERR = ,F20.15)
C
      FACT = 8.0*PI* EXP(-2.0*GDOU2)
      H = 1.0/ SQRT(2.0*GDOU2)
C
      DO 7 I=1,NUMINT
C
      READ (5,21) N, M, NP, MP
   21 FORMAT (4I10)
      INDNUM(I) = 1000*N+100*M+10*NP+MP
      IND(I) = N+M+NP+MP
C
      IF (IND(I)/2*2-IND(I)) 25,26,25
   26 SINT(I) = 0.0
      GO TO 7
C
   25 WRITE (6,61) N, M, NP, MP
   61 FORMAT (///1H0,4HN = ,I2,10X,4HM = ,I2,10X,5HNP = ,I2,
     110X,5HMP = ,I2)
      A = 0.0
      B =  SQRT(10.0)*H
      SUMS = 0.0
      ERUSED = ERR
C
      DO 14 K=1,LIMIT1
C
```

```
   50 CALL SMPSN(F, A, B, ERUSED, LIMIT2, SI1, S, NUM, IER)
      WRITE (6,201) SI1, S, NUM, IER, ERUSED
  201 FORMAT (1H0,6HSI1 = ,F20.15,10X,4HS = ,F20.15,10X,6HNUM = ,
     1I5,10X,6HIER = ,I5/1H0,9HERUSED = ,F20.15)
      IF (IER-4) 5,6,5
    6 ERUSED = ERUSED*10.0
      GO TO 50
C
    5 SUMS = SUMS + S
      IF ( ABS(S)- ABS(ERR*SUMS)) 13,13,144
  144 A = B
      AKP1 = K+1
   14 B =  SQRT(AKP1*10)*H
C
      WRITE (6,15)
   15 FORMAT (1H0,41HREQUIRED ACCURACY NOT MET IN LIMIT1 STEPS)
C
   13 SINT(I) = SUMS*FACT
      WRITE (6,91) SINT(I), ERUSED
   91 FORMAT (1H0,7HSINT = ,F20.15,10X,9HERUSED = ,F20.15)
C
    7 CONTINUE
C
      WRITE (6,499) (SINT(I), INDNUM(I), I=1,NUMINT)
  499 FORMAT (1H1,17X,4HSINT,25X,4HNMNM///(1H0,10X,F20.15,10X,
     1I10))
      CALL EXIT
      END
C
C.................................................................
C
C     FUNCTION F(X)
C
C.................................................................
C
      FUNCTION F(X)
C
      COMMON N, M, NP, MP, GDOU2, GCOU2, ERRBES, LIMBES
      DOUBLE PRECISION BESSJ, ER, ARG, DEXP, GDOU2D, X2D
C
      X2 = X**2
      RC =  SQRT(1.0+X2)
      RM = (RC+X)**M
      RMP = (RC+X)**MP
      ARG = GCOU2*RC
      N1 = N
      N2 = NP
      ER = ERRBES
      LIM = LIMBES
```

```
      GDOU2D = GDOU2
      X2D = X2
      F = 0.25*(RM+1.0/RM)*(RMP+1.0/RMP)*BESSJ(N1,ARG,ER,LIM)*
     1BESSJ(N2,ARG,ER,LIM)/RC*DEXP(-2.0*GDOU2D*X2D)
      RETURN
      END
```

```
C
C.....................................................................
C
C     EVALUATION OF DOUBLE INTEGRAL BY SIMPSON S RULE
C
C.....................................................................
C
      EXTERNAL G
      DIMENSION DINT(1000), IND(1000), INDNUM(1000)
      COMMON N, M, NP, MP, GDOU2, GCOU2, ERRBES, LIMBES, HODER,
     1ERRDER, LIMDER, ERR, LIMIT1, LIMIT2, KO, FP, RC, NTMTB
C
      PI = 3.141593
      NTMTB = 0
C
      READ (5,11) ERR, ERRBES, LIMIT1, LIMIT2, LIMBES, NUMINT
   11 FORMAT (2F20.10,3I10/I10)
      READ (5,47) HODER, ERRDER, LIMDER
   47 FORMAT (F10.5,F20.10,I10)
      READ (5,1) FR, DOC
    1 FORMAT (2F10.5)
C
      GCOU2 = 1.0/(2.0*FR**2)
      GDOU2 = GCOU2*DOC
      WRITE (6,2) FR, GCOU2, GDOU2, DOC, ERR
    2 FORMAT (1H1,21HFR = U/SQRT(2*G*C) = ,F10.5,10X,
     111HG*C/U**2 = ,F10.5,10X,11HG*D/U**2 = ,F10.5,10X,
     26HD/C = ,F10.5,10X/1H0,6HERR = ,F20.15)
      WRITE (6,701) LIMIT1, LIMIT2, ERRBES, LIMBES, HODER,
     1ERRDER, LIMDER, NUMINT
  701  FORMAT (1H0,9HLIMIT1 = ,I5,10X,9HLIMIT2 = ,I5/1H0,
     19HERRBES = ,F20.15,10X,9HLIMBES = ,I5/1H0,8HHODER = ,
     2F10.5,10X,9HERRDER = ,F20.15,10X,9HLIMDER = ,I5/1H0,
     39HNUMINT = ,I5)
C
      DO 7 I=1,NUMINT
C
      READ (5,21) N, M, NP, MP
   21 FORMAT (4I10)
      INDNUM(I) = 1000*N+100*M+10*NP+MP
      IND(I) = N+M+NP+MP
C
      IF (IND(I)/2*2-IND(I)) 26,25,26
   26 DINT(I) = 0.0
      GO TO 7
C
   25 WRITE (6,61) N, M, NP, MP
   61 FORMAT(////1H0,4HN = ,I2,10X,4HM = ,I2,10X,5HNP = ,I2,
     110X,5HMP = ,I2//)
```

```
      A = 0.0
      B = 5.0
      SUMS = 0.0
      ERUSED = 10.0*ERR
C
      DO 14 L=1,LIMIT1
   50 CALL SMPSN1(G, A, B, ERUSED, LIMIT2, SI1, S, NUM, IER)
      WRITE (6,201) SI1, S, NUM, IER, ERUSED, A, B
  201 FORMAT (////1H0,6HSI1 = ,F20.15,10X,4HS = ,F20.15,10X,
     16HNUM = ,I5,10X,6HIER = ,I5/1H0,9HERUSED = ,F20.15,10X,
     24HA = ,F20.15,10X,4HB = ,F20.15////)
      IF (IER-4) 5,6,5
    6 ERUSED = ERUSED*10.0
      IF (ERUSED.GE.0.1) CALL EXIT
      GO TO 50
    5 SUMS = SUMS + S
      IF ( ABS(S)- ABS(ERR*SUMS)) 13,13,144
  144 A = B
   14 B = B + 20.0
C
      BB = B - 20.0
      WRITE (6,15) BB
   15 FORMAT (1H0,41HREQUIRED ACCURACY NOT MET IN LIMIT1 STEPS,
     110X,4HB = ,F20.15)
C
   13 DINT(I) = SUMS*GCOU2
      WRITE (6,91) DINT(I), ERUSED, B, SUMS
   91 FORMAT (//////1H0,7HDINT = ,F20.15,10X,9HERUSED = ,
     1F20.15,10X,4HB = ,F20.10/1H0,7HSUMS = ,F20.15)
C
    7 CONTINUE
C
      WRITE (6,499) (DINT(I), INDNUM(I), I=1,NUMINT)
  499 FORMAT (1H1,17X,4HDINT,25X,4HNMNM///(1H0,10X,F20.15,10X,
     1I10))
      WRITE (7,599) (DINT(I), INDNUM(I), I=1,NUMINT)
  599 FORMAT (E20.10, 20X, I10)
C
      CALL EXIT
      END
C
C.....................................................................
C
C     FUNCTION G(U)
C
C.....................................................................
C
      FUNCTION G(U)
      EXTERNAL FM
```

```
      REAL KO, K
      COMMON N, M, NP, MP, GDOU2, GCOU2, ERRBES, LIMBES, HODER,
     1ERRDER, LIMDER, ERR, LIMIT1, LIMIT2, KO, FP, RC, NTMTB
C
C     THE DERIVATIVE OF THE FUNCTION AT KO = 1.0+U**2 IS
C     EVALUATED BY DIFFERENTIATING AN INTERPOLATION POLYNOMIAL
C     OF THE FOURTH DEGREE
C
      KO = 1.0+U**2
      RC =  SQRT(KO)
      ERDER = ERRDER
    3 H = HODER
      PFP = 0.0
      DO 10 J=1,LIMDER
      H = H/2.0
      FP = (F(KO-2.0*H)-8.0*F(KO-H)+8.0*F(KO+H)-F(KO+2.0*H))
     1/(12.0*H)
      IF ( ABS(FP-PFP)- ABS(FP*ERDER)) 4,4,10
   10 PFP = FP
C
      WRITE (6,60) U, FP, H
   60 FORMAT (////1HO,41HREQUIRED ACCURACY NOT MET IN LIMDER STEPS,
     15X,4HU = ,F20.15,5X,5HFP = ,F20.15,5X,4HH = ,F20.15)
      IF ( ABS(FP).LE.1.0E-14) GO TO 4
      ERDER = ERDER*10.0
      IF (ERDER.GT.0.1) CALL EXIT
      GO TO 3
C
    4 SUMS = 0.0
      ERUSED = ERR
      A = 0.0
      B = 0.0
      U2 = U**2
      DO 90 I = 1,LIMIT1
      A = B
      B = B + 2.5/GDOU2
      IF (B.GE.U2) B=U2
   91 CALL SMPSN (FM, A, B, ERUSED, LIMIT2, SI1, S, NUM, IER)
      IF (IER-4) 77,88,77
   88 ERUSED = ERUSED*10.0
      IF (ERUSED.GE.0.1) CALL EXIT
      GO TO 91
   77 SUMS = SUMS + S
      IF (B-U2) 96,92,92
   96 IF ( ABS(S)- ABS(ERR*SUMS)) 97,97,90
   90 CONTINUE
C
      WRITE (6,93) U
   93 FORMAT (1HO,32HKMAX GREATER THAN 2*LIMIT1/GDOU2,10X,
```

```
     14HU = ,F20.15)
      CALL EXIT
C
   97 A = U2
      B = 2.0 + U2
      ERUSED = ERR
      IMAX = 5
      CALL SMPSN (FM, A, B, ERUSED, IMAX, SI1, PVAL, NUM, IER)
      IF ( ABS(PVAL)- ABS(ERR*SUMS)) 13,92.92
C
   92 A = U2
      B = 2.0 + U2
      ERUSED = ERR
   94 CALL SMPSN (FM, A, B, ERUSED, LIMIT2, SI1, PVAL, NUM, IER)
      IF (IER-4) 55,66,55
   66 IF ( ABS(PVAL)- ABS(ERR*SUMS)) 55,55,606
  606 ERUSED = ERUSED*10.0
      IF (ERUSED.GE.0.1) CALLEXIT
      GO TO 94
   55 SUMS = SUMS + PVAL
C
      A = B
      B = B + 2.0/GDOU2
      ERUSED = ERR
      DO 14 L=1,LIMIT1
   50 CALL SMPSN (FM, A, B, ERUSED, LIMIT2, SI1, S, NUM, IER)
      IF (IER-4) 5,6,5
    6 IF ( ABS(S)- ABS(ERR*SUMS)) 5,5,6666
 6666 ERUSED = ERUSED*10.0
      IF (ERUSED.GE.0.1) CALL EXIT
      GO TO 50
    5 SUMS = SUMS + S
      IF ( ABS(S)- ABS(ERR*SUMS)) 13,13,144
  144 A = B
   14 B = B + 2.0/GDOU2
C
      WRITE (6,15) U, B
   15 FORMAT (1H0,41HREQUIRED ACCURACY NOT MET IN LIMIT1 STEPS,
     110X,4HU = ,F20.15,10X,4HB = ,F20.15)
      NTMTB = NTMTB + 1
      IF (NTMTB.GT.50) CALL EXIT
C
   13 RM = (1.0+U/RC)**M
      RMP = (1.0+U/RC)**MP
      G = (RM+1.0/(RM*(KO**M)))*(RMP+1.0/(RMP*(KO**MP)))/
     1(RC**(N+NP+2-M-MP))*SUMS
      WRITE (6,301) SUMS, U, G, B
  301 FORMAT (1H0,7HSUMS = ,F20.15,10X,4HU = ,F20.15,10X,
     14HG = ,F20.15,10X,7HKMAX = ,F20.10)
```

```
      RETURN
      END
C
C.................................................................
C
C     FUNCTION FM(K)
C
C.................................................................
C
      FUNCTION FM(K)
      REAL KO, K
      COMMON N, M, NP, MP, GDOU2, GCOU2, ERRBES, LIMBES, HODER,
     1ERRDER, LIMDER, ERR, LIMIT1, LIMIT2, KO, FP, RC
C
      IF (K-KO) 3,2,3
    3 FM = F(K)/(K-KO)
      RETURN
    2 FM = FP
      RETURN
      END
C
C.................................................................
C
C     FUNCTION F(K)
C
C.................................................................
C
      FUNCTION F(K)
      REAL KO, K
      COMMON N, M, NP, MP, GDCU2, GCOU2, ERRBES, LIMBES, HODER,
     1ERRDER, LIMDER, ERR, LIMIT1, LIMIT2, KO, FP, RC
      DOUBLE PRECISION BESSJM, ER, ARG, DEXP, GDOU2D, KD
C
      ARG = GCOU2*K/RC
      N1 = N
      N2 = NP
      ER = ERRBES
      LIM = LIMBES
      GDOU2D = GDOU2
      KD = K
      F = (K+KO)*DEXP(-2.0*KD*GDOU2D)*((GCOU2*K)**(N+NP))*
     1BESSJM(N1,ARG,ER,LIM)*BESSJM(N2,ARG,ER,LIM)
      RETURN
      END
```

```
C
C.................................................................
C
C     FUNCTION BESSJM
C
C     BESSEL FUNCTION J OF ORDER  N+1/2  DIVIDED BY THE
C     ARGUMENT RAISED TO THE  N+1/2  POWER
C
C.................................................................
C
      DOUBLE PRECISION FUNCTION BESSJM(N,X,ERR,LIMIT)
      DOUBLE PRECISION X, ERR, PI, FACTOR, TERM, SUM, X2, DSIN,
     1DCOS, DSQRT, DABS, BESJ, FACT
      DIMENSION BESJ(100)
      PI = 3.1415926535898
      IF (X-5.0) 2,2,3
C
    3 NP1 = N+1
C
C     THE RECURRENCE RELATION
C                 J(N+1,X)+J(N-1,X) = (2*N/X)*J(N,X)
C     IS USED
C
      IF (N-1) 13,11,16
   13 BESJ(1) = DSIN(X)
      GO TO 18
   11 BESJ(2) = DSIN(X)/X-DCOS(X)
      GO TO 18
   16 BESJ(1) = DSIN(X)
      BESJ(2) = DSIN(X)/X-DCOS(X)
      DO 19 I=3,NP1
      AI = I
   19 BESJ(I) = (2.0*AI-3.0)/X*BESJ(I-1)-BESJ(I-2)
   18 BESSJM = BESJ(NP1)*DSQRT(2.0/PI)/(X**(N+1))
      GO TO 101
C
    2 FACT = 1.0
C
C     THE EXPRESSION OF A BESSEL FUNCTION J AS A SERIES OF
C     ASCENDING POWERS OF THE ARGUMENT IS USED
C
      IF (N) 8,7,8
    8 DO 5 I = 1,N
      A2IP1 = 2*I+1
    5 FACT = FACT/A2IP1
    7 FACTOR = DSQRT(2.0/PI)*FACT
      TERM = 1.0
      SUM = 1.0
      AN2 = 2*N
```

```
      X2 = X**2
      DO 10 I=1,LIMIT
      AI2 = 2*I
      TERM = (-1.0)*TERM*X2/(AI2*(AN2+AI2+1.0))
      SUM = SUM+TERM
      IF (DABS(TERM)-DABS(ERR*SUM)) 12,12,10
   10 CONTINUE
      WRITE (6,113) TERM, SUM, X
  113 FORMAT (1H0,40HREQUIRED ACCURACY NOT MET IN LIMIT STEPS/1H ,
     17HTERM = ,F20.15,10X,6HSUM = ,F20.15,10X,4HX = ,F20.15)
   12 BESSJM = SUM*FACTOR
  101 RETURN
      END
```

```
C
C..................................................................
C
C     SOLUTION OF THE SYSTEM OF EQUATIONS
C
C                               (I-B)A = R
C
C..................................................................
C
      DIMENSION AINT(20,20), D(20,20), IND(20,20), INDNUM(20,20)
      DIMENSION N(20,20), M(20,20), NP(20,20), MP(20,20)
      DIMENSION B(20,20), BB(20,20), BCOL(400), R(20)
C
      READ (5,1) EPS
    1 FORMAT (F20.10)
C
      READ (5,2) NUMAOC, NUMSYS, MNUMEQ
    2 FORMAT (3I10)
C
      READ (5,61) ((AINT(I,J),N(I,J),M(I,J),NP(I,J),MP(I,J),
     1J=1,I),I=1,MNUMEQ)
   61 FORMAT (E20.10,26X,4I1)
C
      WRITE (6,991) ((AINT(I,J),N(I,J),M(I,J),NP(I,J),MP(I,J),
     1J=1,I),I=1,MNUMEQ)
  991 FORMAT (1H1/(1H0,F20.10,10X,4I1))
C
      MNEQM1 = MNUMEQ-1
      DO 10 I=1,MNEQM1
C
      IP1 = I+1
C
      DO 10 J=IP1,MNUMEQ
C
      AINT(I,J) = AINT(J,I)
      N(I,J) = NP(J,I)
      M(I,J) = MP(J,I)
      NP(I,J) = N(J,I)
      MP(I,J) = M(J,I)
C
   10 CONTINUE
C
      DO 20 I=1,MNUMEQ
C
      DO 20 J=1,MNUMEQ
C
      IND(I,J) = N(I,J)+M(I,J)+NP(I,J)+MP(I,J)
      INDNUM(I,J) = 1000*N(I,J)+100*M(I,J)+10*NP(I,J)+MP(I,J)
C
```

```
      IF ((IND(I,J)/2*2-IND(I,J)) 25,26,25
C
   26    D(I,J) = AINT(I,J)*(-1)**(IND(I,J)/2+NP(I,J))
      GO TO 20
C
   25    D(I,J) = AINT(I,J)*(-1)**((IND(I,J)+1)/2+NP(I,J))
C
   20 CONTINUE
C
      WRITE (6,662)
  662 FORMAT(1H1,11HMATRIX AINT////)
      WRITE (6,62) ((AINT(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   62 FORMAT (1H0,9E14.6/1H0,9E14.6/1H0,98X,2E14.6)
C
      WRITE (6,31)
   31 FORMAT (1H1,8HMATRIX D////)
      WRITE (6,13) ((D(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   13 FORMAT (1H0,9E14.6/1H0,9E14.6/1H0,98X,2E14.6)
C
      DO 501 K=1,NUMAOC
C
      READ (5,213) AOCM1
  213 FORMAT (F20.10)
      AOC = AOCM1+1.0
      WRITE (6,41) AOC
   41 FORMAT (1H1,6HA/C = ,F10.8)
C
      CALL MATRIX(D,BB,QDOT1,MNUMEQ,AOCM1)
C
      WRITE (6,73) ((BB(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   73 FORMAT (////1H0,8HMATRIX B////(1H0,9E14.6)/1H0,9E14.6/
     11H0,98X,2E14.6)
C
      DO 515 I=1,MNUMEQ
      DO 515 J=1,MNUMEQ
  515 B(I,J) = -1.0*BB(I,J)
C
      DO 51 I=1,MNUMEQ
   51 B(I,I) = 1.0-BB(I,I)
C
      WRITE (6,74) ((B(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   74 FORMAT (////1H0,12HMATRIX (I-B)////(1H0,9E14.6)/1H0,
     19E14.6/1H0,2E14.6)
C
      WRITE (6,14) QDOT1
   14 FORMAT (////1H0,8HQDOT1 = ,E14.6)
      R(1) = -1.0/QDOT1
      WRITE (6,99) R(1)
   99 FORMAT (1H1,7HR(1) = ,E14.6)
```

```
C
      NUMEQ = 0
      DO 501 KK=1,NUMSYS
C
      NUMEQ = NUMEQ + (KK+1)
C
      DO 2222 I=1,NUMEQ
 2222 R(I) = 0.0
      R(1) = -1.0/QDOT1
C
      DO 97 J=1,NUMEQ
      DO 97 I=1,NUMEQ
      L = (J-1)*NUMEQ+I
   97 BCOL(L) = B(I,J)
C
      CALL GELG(R,BCOL,NUMEQ,1,EPS,IER)
C
      WRITE (6,16) IER
   16 FORMAT (//////1HO,6HIER = ,I5)
      WRITE (6,17) (R(I),I=1,NUMEQ)
   17 FORMAT (//1HO,6HA10 = ,E14.6,5X,6HA11 = ,E14.6/1HO,6HA20 = ,
     1E14.6,5X,6HA21 = ,E14.6,5X,6HA22 = ,E14.6/1HO,6HA30 = ,E14.6,
     1 5X,6HA31 = ,E14.6, 5X,6HA32 = ,E14.6, 5X,6HA33 = ,E14.6/1HO,
     16HA40 = ,E14.6, 5X,6HA41 = ,E14.6, 5X,6HA42 = ,E14.6, 5X,
     16HA43 = ,E14.6, 5X,6HA44 = ,E14.6/1HO,6HA50 = ,E14.6, 5X,
     16HA51 = ,E14.6, 5X,6HA52 = ,E14.6, 5X,6HA53 = ,E14.6, 5X,
     16HA54 = ,E14.6/1HO,100X,6HA55 = ,E14.6)
C
      WRITE (7,717) (R(I), I, KK, AOC, I=1,NUMEQ)
  717 FORMAT (E20.10,I10,20X,I10,10X,F10.6)
C
  501 CONTINUE
C
      CALL EXIT
      END
C
C..................................................................
C
C     SUBROUTINE MATRIX(D,B,QDOT1,NUMEQ,AOCM1)
C
C     PURPOSE
C
C     OBTAIN THE AUGMENTED MATRIX OF THE SYSTEM OF EQUATIONS
C
C                              (I-B)A = C
C
C..................................................................
C
      SUBROUTINE MATRIX(D,B,QDOT1,NUMEQ,AOCM1)
```

```
C
      DIMENSION D(20,20), B(20,20)
      DIMENSION POL(10), DPOL(10), QOL(10), DQOL(10), P(10,10),
     1DP(10,10), Q(10,10), DQ(10,10), DPDQ(10,10), DPLDQL(10)
      DOUBLE PRECISION QO, POL, DPOL, QOL, DQOL, P, DP, Q, DQ, XM1,
     1DPDQ, DPLDQL
C
      XM1 = AOCM1
      CALL LEGF(QO,POL,DPOL,QOL,DQOL,P,DP,Q,DQ,XM1)
C
      NDEG = 5
C
      WRITE (6,32)
   32 FORMAT (////1HO,11HFUNCTIONS P)
      WRITE (6,11) (POL(I),(P(I,J),J=1,I),I=1,NDEG)
C
      WRITE (6,33)
   33 FORMAT   (//1HO,11HFUNCTIONS Q)
      WRITE (6,11) (QOL(I),(Q(I,J),J=1,I),I=1,NDEG)
C
      WRITE (6,34)
   34 FORMAT   (//1HO,30HDERIVATIVES OF THE FUNCTIONS P)
      WRITE (6,11) (DPOL(I),(DP(I,J),J=1,I),I=1,NDEG)
C
      WRITE (6,35)
   35 FORMAT   (//1HO,30HDERIVATIVES OF THE FUNCTIONS Q)
      WRITE (6,11) (DQOL(I),(DQ(I,J),J=1,I),I=1,NDEG)
C
   11 FORMAT (1HO,2F25.15/1HO,3F25.15/1HO,4F25.15/(1HO,5F25.15))
C
      DO 87 I=1,NDEG
      DPLDQL(I) = DPOL(I)/DQOL(I)
      DO 87 J=1,I
   87 DPDQ(I,J) = DP(I,J)/DQ(I,J)
C
      WRITE (6,36)
   36 FORMAT   (//1HO,20HPDOT DIVIDED BY QDOT)
      WRITE (6,11) (DPLDQL(I),(DPDQ(I,J),J=1,I),I=1,NDEG)
C
      DO 50 J=1,NUMEQ
      B(1,J) = DPLDQL(1)*D(1,J)*3.0/4.0
      B(2,J) = DPDQ(1,1)*D(2,J)*3.0/2.0
      B(3,J) = DPLDQL(2)*D(3,J)*5.0/4.0
      B(4,J) = DPDQ(2,1)*D(4,J)*5.0/2.0
      B(5,J) = DPDQ(2,2)*D(5,J)*5.0/2.0
      B(6,J) = DPLDQL(3)*D(6,J)*7.0/4.0
      B(7,J) = DPDQ(3,1)*D(7,J)*7.0/2.0
      B(8,J) = DPDQ(3,2)*D(8,J)*7.0/2.0
      B(9,J) = DPDQ(3,3)*D(9,J)*7.0/2.0
```

```
      B(10,J) = DPLDQL(4)*D(10,J)*9.0/4.0
      B(11,J) = DPDQ(4,1)*D(11,J)*9.0/2.0
      B(12,J) = DPDQ(4,2)*D(12,J)*9.0/2.0
      B(13,J) = DPDQ(4,3)*D(13,J)*9.0/2.0
      B(14,J) = DPDQ(4,4)*D(14,J)*9.0/2.0
      B(15,J) = DPLDQL(5)*D(15,J)*11.0/4.0
      B(16,J) = DPDQ(5,1)*D(16,J)*11.0/2.0
      B(17,J) = DPDQ(5,2)*D(17,J)*11.0/2.0
      B(18,J) = DPDQ(5,3)*D(18,J)*11.0/2.0
      B(19,J) = DPDQ(5,4)*D(19,J)*11.0/2.0
      B(20,J) = DPDQ(5,5)*D(20,J)*11.0/2.0
C
   50 CONTINUE
C
      QDOT1 = DQOL(1)
C
      RETURN
      END
C
C.................................................................
C
C     SUBROUTINE LEGF(QO,POL,DPOL,QOL,DQOL,P,DP,Q,DQ,XM1)
C
C     PURPOSE
C
C     EVALUATE THE LEGENDRE FUNCTIONS AND THEIR FIRST DERIVATIVES
C     FOR VALUES OF THE ARGUMENT GREATER THAN 1 AND DEGREES UP TO 5
C
C.................................................................
C
      SUBROUTINE LEGF(QO,POL,DPOL,QOL,DQOL,P,DP,Q,DQ,XM1)
C
      DIMENSION POL(10), DPOL(10), QOL(10), DQOL(10), P(10,10),
     1DP(10,10), Q(10,10), DQ(10,10)
      DOUBLE PRECISION QO, POL, DPOL, QOL, DQOL, P, DP, Q, DQ,
     1X, XM1, X2M1, X2M12, X2M13, RC, RC3, RC5, RC7, DLOG, DSQRT
C
      X = 1.0+XM1
      X2M1 = XM1*(2.0+XM1)
      X2M12 = X2M1*X2M1
      X2M13 = X2M12*X2M1
      RC = DSQRT(X2M1)
      RC3 = X2M1*RC
      RC5 = X2M12*RC
      RC7 = X2M13*RC
C
      POL(1) = X
      POL(2) = 1.5*X2M1+1.0
      POL(3) = 2.5*X*X2M1+X
```

```
      POL(4) = 4.375*X2M12+5.0*X2M1+1.0
      POL(5) = X*(7.875*X2M12+7.0*X2M1+1.0)
C
      DPOL(1) = 1.0
      DPOL(2) = 3.0*X
      DPOL(3) = 7.5*X2M1+6.0
      DPOL(4) = X*(17.5*X2M1+10.0)
      DPOL(5) = 315.0/8.0*X2M12+52.5*X2M1+15.0
C
      P(1,1) = RC
      P(2,1) = 3.0*X*RC
      P(2,2) = 3.0*X2M1
      P(3,1) = 7.5*RC3+6.0*RC
      P(3,2) = 15.0*X*X2M1
      P(3,3) = 15.0*RC3
C
      P(4,1) = X*(17.5*RC3+10.0*RC)
      P(4,2) = 7.5*(7.0*X2M12+6.0*X2M1)
      P(4,3) = 105.0*X*RC3
      P(4,4) = 105.0*X2M12
C
      P(5,1) = 315.0/8.0*RC5+52.5*RC3+15.0*RC
      P(5,2) = 52.5*X*(3.0*X2M12+2.0*X2M1)
      P(5,3) = 52.5*(9.0*RC5+8.0*RC3)
      P(5,4) = 945.0*X*X2M12
      P(5,5) = 945.0*RC5
C
      DP(1,1) = X/RC
      DP(2,1) = 6.0*RC+3.0/RC
      DP(2,2) = 6.0*X
      DP(3,1) = 1.5*X*(15.0*RC+4.0/RC)
      DP(3,2) = 45.0*X2M1+30.0
      DP(3,3) = 45.0*X*RC
C
      DP(4,1) = 70.0*RC3+72.5*RC+10.0/RC
      DP(4,2) = 30.0*X*(7.0*X2M1+3.0)
      DP(4,3) = 105.0*(4.0*RC3+3.0*RC)
      DP(4,4) = 420.0*X*X2M1
C
      DP(5,1) = 15.0/8.0*X*(105.0*RC3+84.0*RC+8.0/RC)
      DP(5,2) = 52.5*(15.0*X2M12+18.0*X2M1+4.0)
      DP(5,3) = 52.5*X*(45.0*RC3+24.0*RC)
      DP(5,4) = 945.0*(5.0*X2M12+4.0*X2M1)
      DP(5,5) = 4725.0*X*RC3
C
      Q0 = 0.5*(DLOG(2.0+XM1)-DLOG(XM1))
C
      QOL(1) = X*Q0-1.0
      QOL(2) = (1.5*X2M1+1.0)*Q0-1.5*X
```

```
      QOL(3) = X*(2.5*X2M1+1.0)*Q0-2.5*X2M1-11.0D0/6.0D0
      QOL(4) = (4.375*X2M12+5.0*X2M1+1.0)*Q0-X*(4.375*X2M1+
     125.0D0/12.0D0)
      QOL(5) = X*(7.875*X2M12+7.0*X2M1+1.0)*Q0-7.875*X2M12-
     19.625*X2M1-137.0D0/60.0D0
C
      DQOL(1) = Q0-X/X2M1
      DQOL(2) = 3.0*X*Q0-3.0-1.0/X2M1
      DQOL(3) = (7.5*X2M1+6.0)*Q0-7.5*X-X/X2M1
      DQOL(4) = 2.5*X*(7.0*X2M1+4.0)*Q0-17.5*X2M1-95.0D0/6.0D0-
     11.0/X2M1
      DQOL(5) = 15.0/8.0*(21.0*X2M12+28.0*X2M1+8.0)*Q0-105.0/8.0*
     1X*(3.0*X2M1+2.0)-X/X2M1
C
      Q(1,1) = Q0*RC-X/RC
      Q(2,1) = 3.0*X*RC*Q0-3.0*RC-1.0/RC
      Q(2,2) = 3.0* X2M1*Q0-(3.0*X**3-5.0*X)/X2M1
      Q(3,1) = (7.5*X2M1+6.0)*RC*Q0-7.5*X*RC-X/RC
      Q(3,2) = 15.0*X*Q0*X2M1-15.0*X2M1-5.0+2.0/X2M1
      Q(3,3) = 15.0*RC3*Q0-X*(15.0*RC-10.0/RC+8.0/RC3)
C
      Q(4,1) = X*(17.5*X2M1+10.0)*RC*Q0-17.5*RC3-95.0D0/6.0D0*
     1RC-1.0/RC
      Q(4,2) = (52.5*X2M12+45.0*X2M1)*Q0-52.5*X*X2M1-10.0*X+
     12.0*X/X2M1
      Q(4,3) = 105.0*X*RC3*Q0-105.0*RC3-35.0*RC+14.0/RC-8.0/RC3
      Q(4,4) = 105.0*X2M12*Q0-X*(105.0*X2M1-70.0+56.0/X2M1-48.0/
     1X2M12)
C
      Q(5,1) = 15.0/8.0*(21.0*X2M12+28.0*X2M1+8.0)*RC*Q0-X*(
     1315.0/8.0*RC3+26.25*RC+1.0/RC)
      Q(5,2) = 52.5*X*(3.0*X2M12+2.0*X2M1)*Q0-157.5*(X2M12+X2M1)-
     114.0+2.0/X2M1
      Q(5,3) = 52.5*(9.0*X2M1+8.0)*RC3*Q0-472.5*X*RC3-105.0*X*
     1RC+28.0*X/RC-8.0*X/RC3
      Q(5,4) = 945.0*X*X2M12*Q0-945.0*X2M12-315.0*X2M1+126.0-
     172.0/X2M1+48.0/X2M12
      Q(5,5) = 945.0*RC5*Q0-X*(945.0*RC3-630.0*RC+504.0/RC-
     1432.0/RC3+384.0/RC5)
C
      DQ(1,1) = X/RC*Q0-1.0/RC+1.0/RC3
      DQ(2,1) = (6.0*RC+3.0/RC)*Q0-6.0*X/RC+X/RC3
      DQ(2,2) = 6.0*X*Q0-6.0-2.0/X2M1-4.0/X2M12
      DQ(3,1) = X*(22.5*RC+6.0/RC)*Q0-22.5*RC-13.5/RC+1.0/RC3
      DQ(3,2) = (45.0*X2M1+30.0)*Q0-X*(45.0+4.0/X2M12)
      DQ(3,3) = 45.0*X*RC*Q0-45.0*RC-15.0/RC+6.0/RC3+24.0/RC5
C
      DQ(4,1) = (70.0*RC3+72.5*RC+10.0/RC)*Q0-X*(70.0*RC+
     1155.0D0/6.0D0/RC-1.0/RC3)
```

```
      DQ(4,2) = 30.0*X*(7.0*X2M1+3.0)*Q0-210.0*X2M1-160.0-2.0/
     1X2M1-4.0/X2M12
      DQ(4,3) = 105.0*(4.0*X2M1+3.0)*RC*Q0-X*(420.0*RC+35.0/RC+
     114.0/RC3-24.0/RC5)
      DQ(4,4) = 420.0*X*X2M1*Q0-140.0-420.0*X2M1+56.0/X2M1-
     132.0/X2M12-192.0/X2M13
C
      DQ(5,1) = 15.0*X*(105.0/8.0*X2M1+10.5+1.0/X2M1)*RC*Q0-
     11575.0/8.0*RC3-1785.0/8.0*RC-165.0/4.0/RC+1.0/RC3
      DQ(5,2) = 52.5*(15.0*X2M12+18.0*X2M1+4.0)*Q0-787.5*X*X2M1-
     1420.0*X-4.0*X/X2M12
      DQ(5,3) = 157.5*X*(15.0*X2M1+8.0)*RC*Q0-2362.5*RC3-2047.5*
     1RC-105.0/RC-12.0/RC3+24.0/RC5
      DQ(5,4) = 945.0*(5.0*X2M12+4.0*X2M1)*Q0-4725.0*X*X2M1-
     1630.0*X+144.0*X/X2M12-192.0*X/X2M13
      DQ(5,5) = 4725.0*X*RC3*Q0-4725.0*RC3-1575.0*RC+630.0/RC-
     1360.0/RC3+240.0/RC5+1920.0/RC7
C
      RETURN
      END
```

```
C
C..........................................................................
C
C     DENSITY OF SOURCE DISTRIBUTION
C
C..........................................................................
C
C
C     PLEASE DRAW WITH BLACK INK
C
      DIMENSION POL1(10), P1(10,10), POL(10), P(10,10)
      DIMENSION A(20,6), N(20), M(20)
      DIMENSION FACTOR(20,181), SIGMAU(181), THETA(181),
     1THEDEG(181)
      DIMENSION DATA(400), X(1000), Y(1000)
      DOUBLE PRECISION XM1, POL1, P1, POL, P
      CALL TRAPS(-1,-1)
C
      CALL PLOTS(DATA,400,7HDENSITY)
C
      READ (5,1) AOCM1
    1 FORMAT (F20.10)
      AOC = AOCM1+1.0
      AOC2M1 = AOCM1*(AOCM1+2.0)
      FAC =  SQRT(AOC2M1)*12.566372
      WRITE (6,31) AOC
   31 FORMAT (1H1,6HA/C = ,F10.8)
C
      READ (5,2) MNUMEQ, NUMSYS, NTHETA, NUMFI
    2 FORMAT (4I10)
      WRITE (6,32) MNUMEQ, NUMSYS, NTHETA, NUMFI
   32 FORMAT (1H0,9HMNUMEQ = ,I3,10X,9HNUMSYS = ,I3,10X,
     19HNTHETA = ,I3,10X,8HNUMFI = ,I3)
C
      I = 0
      NMAX = NUMSYS-1
C
      DO 10 L=1,NMAX
      LP1 = L+1
      DO 10 LL=1,LP1
      I = I+1
      N(I) = L
   10 M(I) = LL-1
C
      WRITE (6,33) (N(I),M(I),I=1,MNUMEQ)
   33 FORMAT (////1H0,4H N M//(1H ,2I2))
C
      XM1 = AOCM1
      CALL PNMXG1(POL,P,XM1)
```

```
C
      DO 17 K=1,NTHETA
C
      NTHEM1 = NTHETA-1
      ANTHM1 = NTHEM1
      KM1 = K-1
      AKM1 = KM1
      THETA(K) = 3.141593/ANTHM1*AKM1
      THEDEG(K) = 180/NTHEM1*KM1
      X(K) = THEDEG(K)
      FACT =  SQRT((AOC+ COS(THETA(K)))*(AOCM1+1.0-
     1 COS(THETA(K))))*FAC
C
      XM1 =  COS(THETA(K))-1.0
      CALL PNMXL1(POL1,P1,XM1)
C
      DO 17 J=1,MNUMEQ
C
      MJ = M(J)
      NJ = N(J)
      IF (MJ.EQ.0) FACTOR(J,K) = -1.0*POL1(NJ)/POL(NJ)/FACT
      SSIGN = (-1)**(MJ+1)
      IF (MJ.NE.0) FACTOR(J,K) = SSIGN*P1(NJ,MJ)/P(NJ,MJ)/FACT
C
   17 CONTINUE
C
      NUMEQ = 0
C
      DO 277 K=1,NUMSYS
C
      NUMEQ = NUMEQ+K
      READ (5,7) (A(I,K),I=1,NUMEQ)
    7 FORMAT (E20.10)
      WRITE (6,77) NUMEQ
   77 FORMAT (1H1,40HCOEFFICIENTS ANM, NUMBER OF EQUATIONS = ,I2)
      WRITE (6,777) (N(I),M(I),A(I,K),I=1,NUMEQ)
  777 FORMAT (///(1H0,1HA,2I2,4H  = ,E20.10))
C
      IF (K.EQ.1) NUMEQ = 0
  277 CONTINUE
C
      N1 = NTHETA
      N2 = N1+1
      N3 = N1+2
      X(N2) = 0.0
      X(N3) = 9.0
      Y(N2) = -0.10
      Y(N3) = 0.02
      WRITE (6,500) X(N2), X(N3), Y(N2), Y(N3), N2, N3
```

```
  500 FORMAT (//////1H0,4F10.5,2I10)
C
      DO 27 KFI=1,NUMFI
C
      NUFIM1 = NUMFI-1
      ANFIM1 = NUFIM1
      KFIM1 = KFI-1
      AKFIM1 = KFIM1
      FIDEG = 180/NUFIM1*KFIM1
      FI = 3.141593*AKFIM1/ANFIM1
      NUMEQ = 0
C
      CALL PLOT(6.0,-10.5,-3)
      CALL PLOT (0.0,0.75,-3)
      CALL AXIS(0.0,0.0,11HANGLE THETA,-11,20.0,0.0,X(N2),X(N3),20.0)
      CALL AXIS(0.0,0.0,7HDENSITY,7,10.0,90.0,Y(N2),Y(N3),20.0)
      CALL SYMBOL(2.0,8.5,0.14,5HFI = ,0.0,5)
      CALL NUMBER(3.0,8.5,0.14,FIDEG,0.0,1)
C
      DO 26 KK=1,NUMSYS
C
      NUMEQ = NUMEQ+KK
C
      DO 211 K=1,NTHETA
      SIGMAU(K) = 0.0
      DO 21 J=1,NUMEQ
      AMJ = M(J)
   21 SIGMAU(K) = SIGMAU(K)+A(J,KK)*FACTOR(J,K)* COS( AMJ*FI)
      Y(K) = SIGMAU(K)
  211 CONTINUE
C
      WRITE (6,22) NUMEQ,FIDEG
   22 FORMAT (1H1,22HNUMBER OF EQUATIONS = ,I2,10X,5HFI = ,F10.5)
      WRITE (6,222) (SIGMAU(K),THETA(K),K=1,NTHETA)
  222 FORMAT (1H0,10X,7HSIGMA/U,23X,5HTHETA//(1H0,F20.10,10X,
     1F20.10))
C
      CALL LINE (X,Y,N1,1,0,0)
C
      IF (KK.EQ.1) NUMEQ = 0
   26 CONTINUE
C
      CALL PLOT(20.0,0.0,-3)
C
   27 CONTINUE
      CALL EXIT
      END
```

```
C
C.......................................................................
C
C     EVALUATION OF THE WAVE RESISTANCE
C
C.......................................................................
C
      DIMENSION AINT(20,20), D(20,20), IND(20,20), INDNUM(20,20)
      DIMENSION N(20,20), M(20,20), NP(20,20), MP(20,20)
      DIMENSION NN(20), MM(20), A(20,6), AA(20,20)
      DIMENSION S(20,20), SUM(20)
C
      READ (5,201) NUMAOC, NUMSYS, MNUMEQ
  201 FORMAT (3I10)
C
      READ (5,61) ((AINT(I,J),N(I,J),M(I,J),NP(I,J),MP(I,J),
     1J=1,I),I=1,MNUMEQ)
   61 FORMAT (E20.10,26X,4I1)
C
      WRITE (6,991) ((AINT(I,J),N(I,J),M(I,J),NP(I,J),MP(I,J),
     1J=1,I),I=1,MNUMEQ)
  991 FORMAT (1H1/(1H0,F20.10,10X,4I1))
C
      MNEQM1 = MNUMEQ-1
      DO 10 I=1,MNEQM1
C
      IP1 = I+1
C
      DO 10 J=IP1,MNUMEQ
C
      AINT(I,J) = AINT(J,I)
      N(I,J) = NP(J,I)
      M(I,J) = MP(J,I)
      NP(I,J) = N(J,I)
      MP(I,J) = M(J,I)
C
   10 CONTINUE
C
      DO 20 I=1,MNUMEQ
C
      DO 20 J=1,MNUMEQ
C
      IND(I,J) = N(I,J)+M(I,J)+NP(I,J)+MP(I,J)
      INDNUM(I,J) = 1000*N(I,J)+100*M(I,J)+10*NP(I,J)+MP(I,J)
C
      IF (IND(I,J)/2*2-IND(I,J)) 25,26,25
C
   26    D(I,J) = AINT(I,J)*(-1)**(IND(I,J)/2+NP(I,J)+M(I,J))
      GO TO 20
```

```
C
   25 D(I,J) = 0.0
C
   20 CONTINUE
C
      WRITE (6,662)
  662 FORMAT(1H1,11HMATRIX AINT////)
      WRITE (6,62) ((AINT(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   62 FORMAT (1H0,9E14.6/1H0,9E14.6/1H0,98X,2E14.6)
C
      WRITE (6,31)
   31 FORMAT (1H1,8HMATRIX D////)
      WRITE (6,13) ((D(I,J),J=1,MNUMEQ),I=1,MNUMEQ)
   13 FORMAT (1H0,9E14.6/1H0,9E14.6/1H0,98X,2E14.6)
C
      I = 0
      NMAX = NUMSYS-1
C
      DO 11 L=1,NMAX
      LP1 = L+1
      DO 11 LL=1,LP1
      I = I+1
      NN(I) = L
   11 MM(I) = LL-1
C
      DO 5555 KAOC = 1,NUMAOC
C
      READ (5,5554) AOCM1
 5554 FORMAT (F10.5)
      AOC = 1.0 + AOCM1
      WRITE (6,5553) AOC
 5553 FORMAT (1H1,6HA/C = ,F10.5)
C
      NUMEQ = 0
C
      DO 277 K=1,NUMSYS
C
      NUMEQ = NUMEQ+K
      READ (5,7) (A(I,K),I=1,NUMEQ)
    7 FORMAT (E20.10)
      WRITE (6,77) NUMEQ
   77 FORMAT (1H1,40HCOEFFICIENTS ANM, NUMBER OF EQUATIONS = ,I2)
      WRITE (6,777) (NN(I),MM(I),A(I,K),I=1,NUMEQ)
  777 FORMAT (///(1H0,1HA,2I2,4H  = ,E20.10))
C
      IF (K.EQ.1) NUMEQ = 0
  277 CONTINUE
C
C
```

```
      NUMEQ = 0
      DO 50 KK=1,NUMSYS
C
      NUMEQ = NUMEQ+KK
      SUM(KK) = 0.0
C
      WRITE (6,102) NUMEQ
  102 FORMAT (1H1,22HNUMBER OF EQUATIONS = ,I2/////1H0,10X,
     14HTERM,22X,4HNMNM)
C
      DO 41 K=1,NUMEQ
      DO 41 I=1,NUMEQ
C
      AA(I,K) = A(I,KK)*A(K,KK)
      IF (D(I,K).EQ.0.0) GO TO 41
      S(I,K) = AA(I,K)*D(I,K)
C
      WRITE (6,101) S(I,K), INDNUM(I,K)
  101 FORMAT (1H0,E20.10,10X,I10)
C
      SUM(KK) = SUM(KK) + S(I,K)
C
   41 CONTINUE
C
      WRITE (6,103) SUM(KK)
  103 FORMAT (////1H0,6HSUM = ,E20.10)
C
      IF (KK.EQ.1) NUMEQ = 0
   50 CONTINUE
C
 5555 CONTINUE
C
      CALL EXIT
      END
```